**Definition:** If $R$ is a commutative ring and $I$ is an ideal in $R$, let $R/I$ represent the set of additive cosets $a + I$ with addition defined by $(a + I) + (b + I) = ((a + b) + I)$, and multiplication defined by $(a + i)(b + I) = (ab + I)$.

**Theorem:** The above definition is well-defined, and yields a ring with additive identity $(0 + I)$ and multiplicative identity $(1 + I)$.

**Theorem:** (First Homomorphism Theorem for Rings) If $f : R_1 \rightarrow R_2$ is a ring homomorphism, then $R_1/\ker f \cong \text{im } f$.

**Theorem:** If $K$ is a field and $I = (p)$ is an ideal with $p$ an irreducible polynomial in $K[x]$, then $K[x]/I$ is a field.

**Example:** $\mathbb{R}[x]/(x^2 + 1)$ is a field, and we can show that it isomorphic to $\mathbb{C}$.

**Theorem:** If $K$ is a field and $p \in K[x]$ is an irreducible polynomial, then the field $K[x]/(p)$ contains an isomorphic copy of $K$ and a root $z = x + I$ of $p$.

**Theorem:** Using the notation from the previous theorem, if $g \in K[x]$ has a root $z$, then $p \mid g$.

**Theorem:** Let $p \in K[x]$ be irreducible of degree $n$, and let $I = (p)$, and let $F = K[x]/I$. Then every element in $F$ has a unique expression of the form
\[b_0 + b_1z + b_2z^2 + \ldots + b_{n-1}z^{n-1}\]
where $z$ is a root of $p$ and the $b_i$’s are in $K$.

**Corollary:** (Converse to the third theorem on this handout.) If $K$ is a field, and $f \in K[x]$, and $K[x]/(f)$ is a field, then $f$ is an irreducible polynomial in $K[x]$.