The Calculus of Functions of Several Variables

Section 4.2
Best Affine Approximations

Best affine approximations
The following definitions should look very familiar.

Definition Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined on an open ball containing the point $c$. We call an affine function $A : \mathbb{R}^m \to \mathbb{R}^n$ the best affine approximation to $f$ at $c$ if (1) $A(c) = f(c)$ and (2) $\|R(h)\|$ is $o(h)$, where

$$R(h) = f(c + h) - A(c + h). \quad (4.2.1)$$

Suppose $A : \mathbb{R}^n \to \mathbb{R}^n$ is the best affine approximation to $f$ at $c$. Then, from our work in Section 1.5, there exists an $n \times m$ matrix $M$ and a vector $b$ in $\mathbb{R}^n$ such that

$$A(x) = Mx + b \quad (4.2.2)$$

for all $x$ in $\mathbb{R}^m$. Moreover, the condition $A(c) = f(c)$ implies $f(c) = Mc + b$, and so $b = f(c) - Mc$. Hence we have

$$A(x) = Mx + f(c) - Mc = M(x - c) + f(c) \quad (4.2.3)$$

for all $x$ in $\mathbb{R}^m$. Thus to find the best affine approximation we need only identify the matrix $M$ in (4.2.3).

Definition Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined on an open ball containing the point $c$. If $f$ has a best affine approximation at $c$, then we say $f$ is differentiable at $c$. Moreover, if the best affine approximation to $f$ at $c$ is given by

$$A(x) = M(x - c) + f(c), \quad (4.2.4)$$

then we call $M$ the derivative of $f$ at $c$ and write $Df(c) = M$.

Now suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ and $A$ is an affine function with $A(c) = f(c)$. Let $f_k$ and $A_k$ be the $k$th coordinate functions of $f$ and $A$, respectively, for $k = 1, 2, \ldots, n$, and let $R$ be the remainder function

$$R(h) = f(c + h) - A(c + h)$$
$$= (f_1(c + h) - A_1(c + h), f_2(c + h) - A_2(c + h), \ldots, f_n(c + h) - A_n(c + h)).$$
Then
\[
\frac{R(h)}{\|h\|} = \left( \frac{f_1(c+h) - A_1(c+h)}{\|h\|}, \frac{f_2(c+h) - A_2(c+h)}{\|h\|}, \ldots, \frac{f_n(c+h) - A_n(c+h)}{\|h\|} \right),
\]
and so
\[
\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0,
\]
that is, \( A \) is the best affine approximation to \( f \) at \( c \), if and only if
\[
\lim_{h \to 0} \frac{f_k(c+h) - A_k(c+h)}{\|h\|} = 0
\]
for \( k = 1, 2, \ldots, n \). But (4.2.6) is the statement that \( A_k \) is the best affine approximation to \( f_k \) at \( c \). In other words, \( A \) is the best affine approximation to \( f \) at \( c \) if and only if \( A_k \) is the best affine approximation to \( f_k \) at \( c \) for \( k = 1, 2, \ldots, n \). This result has many interesting consequences.

**Proposition** If \( f_k : \mathbb{R}^m \to \mathbb{R} \) is the \( k \)th coordinate function of \( f : \mathbb{R}^m \to \mathbb{R}^n \), then \( f \) is differentiable at a point \( c \) if and only if \( f_k \) is differentiable at \( c \) for \( k = 1, 2, \ldots, n \).

**Definition** If \( f_k : \mathbb{R}^m \to \mathbb{R} \) is the \( k \)th coordinate function of \( f : \mathbb{R}^m \to \mathbb{R}^n \), then we say \( f \) is \( C^1 \) on an open set \( U \) if \( f_k \) is \( C^1 \) on \( U \) for \( k = 1, 2, \ldots, n \).

Putting our results in Section 3.3 together with the previous proposition and definition, we have the following basic result.

**Theorem** If \( f : \mathbb{R}^m \to \mathbb{R}^n \) is \( C^1 \) on an open ball containing the point \( c \), then \( f \) is differentiable at \( c \).

Suppose \( f : \mathbb{R}^m \to \mathbb{R}^n \) is differentiable at \( c = (c_1, c_2, \ldots, c_m) \) with best affine approximation \( A \) and \( f_k : \mathbb{R}^m \to \mathbb{R} \) and \( A_k : \mathbb{R}^m \to \mathbb{R} \) are the coordinate functions of \( f \) and \( A \), respectively, for \( k = 1, 2, \ldots, n \). Since \( A_k \) is the best affine approximation to \( f_k \) at \( c \), we know from Section 3.3 that
\[
A_k(x) = \nabla f_k(c) \cdot (x - c) + f_k(c)
\]
for all \( x \) in \( \mathbb{R}^m \). Hence, writing the vectors as column vectors, we have
\[
A(x) = \begin{bmatrix}
A_1(x) \\
A_2(x) \\
\vdots \\
A_n(x)
\end{bmatrix} = \begin{bmatrix}
\nabla f_1(c) \cdot (x - c) + f_1(c) \\
\nabla f_2(c) \cdot (x - c) + f_2(c) \\
\vdots \\
\nabla f_n(c) \cdot (x - c) + f_n(c)
\end{bmatrix}
\]
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\[
\begin{bmatrix}
\frac{\partial}{\partial x_1} f_1(c) & \frac{\partial}{\partial x_2} f_1(c) & \cdots & \frac{\partial}{\partial x_m} f_1(c) \\
\frac{\partial}{\partial x_1} f_2(c) & \frac{\partial}{\partial x_2} f_2(c) & \cdots & \frac{\partial}{\partial x_m} f_2(c) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_1} f_n(c) & \frac{\partial}{\partial x_2} f_n(c) & \cdots & \frac{\partial}{\partial x_m} f_n(c)
\end{bmatrix}
\begin{bmatrix}
x_1 - c_1 \\
x_2 - c_2 \\
\vdots \\
x_m - c_m
\end{bmatrix}
+ \begin{bmatrix}
f_1(c) \\
f_2(c) \\
\vdots \\
f_m(c)
\end{bmatrix}. \quad (4.2.8)
\]

It follows that the \(n \times m\) matrix in (4.2.8) is the derivative of \(f\).

**Theorem** If \(f : \mathbb{R}^m \to \mathbb{R}^n\) is differentiable at a point \(c\), then the derivative of \(f\) at \(c\) is given by

\[
Df(c) = \begin{bmatrix}
\frac{\partial}{\partial x_1} f_1(c) & \frac{\partial}{\partial x_2} f_1(c) & \cdots & \frac{\partial}{\partial x_m} f_1(c) \\
\frac{\partial}{\partial x_1} f_2(c) & \frac{\partial}{\partial x_2} f_2(c) & \cdots & \frac{\partial}{\partial x_m} f_2(c) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_1} f_n(c) & \frac{\partial}{\partial x_2} f_n(c) & \cdots & \frac{\partial}{\partial x_m} f_n(c)
\end{bmatrix}. \quad (4.2.9)
\]

We call the matrix in (4.2.9) the *Jacobian matrix* of \(f\), after the German mathematician Carl Gustav Jacob Jacobi (1804-1851). Note that we have seen this matrix before in our discussion of change of variables in integrals in Section 3.7.

**Example** Consider the function \(f : \mathbb{R}^3 \to \mathbb{R}^2\) defined by

\[f(x, y, z) = (xyz, 3x - 2yz).\]

The coordinate functions of \(f\) are

\[f_1(x, y, z) = xyz\]

and

\[f_2(x, y, z) = 3x - 2yz.\]

Now

\[\nabla f_1(x, y, z) = (yz, xz, xy)\]

and

\[\nabla f_2(x, y, z) = (3, -2z, -2y),\]

so the Jacobian of \(f\) is

\[Df(x, y, z) = \begin{bmatrix} yz & xz & xy \\ 3 & -2z & -2y \end{bmatrix}.\]

Hence, for example,

\[Df(1, 2, -1) = \begin{bmatrix} -2 & -1 & 2 \\ 3 & 2 & -4 \end{bmatrix}.\]
Since \( f(1,2,-1) = (-2,7) \), the best affine approximation to \( f \) at \((1,2,-1)\) is
\[
A(x,y,z) = \begin{bmatrix}
-2 & -1 & 2 \\
3 & 2 & -4 \\
\end{bmatrix}
\begin{bmatrix}
x - 1 \\
y - 2 \\
z + 1 \\
\end{bmatrix}
+ \begin{bmatrix}
-2 \\
-7 \\
\end{bmatrix}
= \begin{bmatrix}
-2(x - 1) - (y - 2) + 2(z + 1) - 2 \\
3(x - 1) + 2(y - 2) - 4(z + 1) + 7 \\
-2x - y + 2z + 4 \\
3x + 2y - 4z - 4 \\
\end{bmatrix}.
\]

**Tangent planes**
Suppose \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) parametrizes a surface \( S \) in \( \mathbb{R}^3 \). If \( f_1, f_2, \) and \( f_3 \) are the coordinate functions of \( f \), then the best affine approximation to \( f \) at a point \((s_0,t_0)\) is given by
\[
A(s,t) = \begin{bmatrix}
\frac{\partial}{\partial s} f_1(t_0,s_0) & \frac{\partial}{\partial t} f_1(t_0,s_0) \\
\frac{\partial}{\partial s} f_2(t_0,s_0) & \frac{\partial}{\partial t} f_2(t_0,s_0) \\
\frac{\partial}{\partial s} f_3(t_0,s_0) & \frac{\partial}{\partial t} f_3(t_0,s_0) \\
\end{bmatrix}
\begin{bmatrix}
s - s_0 \\
t - t_0 \\
\end{bmatrix}
+ \begin{bmatrix}
f_1(s_0,t_0) \\
f_2(s_0,t_0) \\
f_3(s_0,t_0) \\
\end{bmatrix}.
\]
(4.2.10)

If the vectors
\[
v = \begin{bmatrix}
\frac{\partial}{\partial s} f_1(s_0,t_0) \\
\frac{\partial}{\partial s} f_2(s_0,t_0) \\
\frac{\partial}{\partial s} f_3(s_0,t_0) \\
\end{bmatrix}
\]
(4.2.11)
and
\[
w = \begin{bmatrix}
\frac{\partial}{\partial t} f_1(s_0,t_0) \\
\frac{\partial}{\partial t} f_2(s_0,t_0) \\
\frac{\partial}{\partial t} f_3(s_0,t_0) \\
\end{bmatrix}
\]
(4.2.12)
are linearly independent, then (4.2.10) implies that the image of \( A \) is a plane in \( \mathbb{R}^3 \) which passes through the point \( f(s_0,t_0) \) on the surface \( S \). Moreover, if we let \( C_1 \) be the curve
on \( S \) through the point \( f(s_0, t_0) \) parametrized by \( \varphi_1(s) = f(s, t_0) \) and \( C_2 \) be the curve on \( S \) through the point \( f(s_0, t_0) \) parametrized by \( \varphi_2(t) = f(s_0, t) \), then \( v \) is tangent to \( C_1 \) at \( f(s_0, t_0) \) and \( w \) is tangent to \( C_2 \) at \( f(s_0, t_0) \). Hence we call the image of \( A \) the tangent plane to the surface \( S \) at the point \( f(s_0, t_0) \).

**Example** Let \( T \) be the torus parametrized by

\[
f(s, t) = ((3 + \cos(t)) \cos(s), (3 + \cos(t)) \sin(s), \sin(t))
\]

for \( 0 \leq s \leq 2\pi \) and \( 0 \leq t \leq 2\pi \). Then

\[
Df(s, t) = \begin{bmatrix}
-3 - \cos(t) & -\sin(t) \\
3 \cos(t) & -\sin(t) \\
0 & \cos(t)
\end{bmatrix}
\]

Thus, for example,

\[
Df\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \begin{bmatrix}
- \left(3 + \frac{1}{\sqrt{2}}\right) & 0 \\
0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

Since

\[
f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(0, 3 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),
\]

the best affine approximation to \( f \) at \( \left(\frac{\pi}{2}, \frac{\pi}{4}\right) \) is

\[
A(s, t) = \begin{bmatrix}
- \left(3 + \frac{1}{\sqrt{2}}\right) & 0 \\
0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
\left(\begin{bmatrix}
s - \frac{\pi}{2} \\
t - \frac{\pi}{4}
\end{bmatrix}
\right) + \begin{bmatrix}
0 & 3 + \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
\left(\begin{bmatrix}
s - \frac{\pi}{2} \\
t - \frac{\pi}{4}
\end{bmatrix}
\right).
\]

Hence

\[
x = - \left(3 + \frac{1}{\sqrt{2}}\right) \left(s - \frac{\pi}{2}\right),
\]

\[
y = -\frac{1}{\sqrt{2}} \left(t - \frac{\pi}{4}\right) + 3 + \frac{1}{\sqrt{2}},
\]

\[
z = \frac{1}{\sqrt{2}} \left(t - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}.
\]
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are parametric equations for the plane \( P \) tangent to \( T \) at \( \left( 0, 3 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \). See Figure 4.2.1.

Chain rule

We are now in a position to state the chain rule in its most general form. Consider functions \( g : \mathbb{R}^m \to \mathbb{R}^q \) and \( f : \mathbb{R}^q \to \mathbb{R}^n \) and suppose \( g \) is differentiable at \( c \) and \( f \) is differentiable at \( g(c) \). Let \( h : \mathbb{R}^m \to \mathbb{R}^n \) be the composition \( h(x) = f(g(x)) \) and denote the coordinate functions of \( f \), \( g \), and \( h \) by \( f_i, g_j, \) and \( h_k \), respectively. Then, for \( k = 1, 2, \ldots, n \),

\[
    h_k(x_1, x_2, \ldots, x_m) = f_k(g_1(x_1, x_2, \ldots, x_m), g_2(x_1, x_2, \ldots, x_m), \ldots, g_q(x_1, x_2, \ldots, x_m)).
\]

Now if we fix \( m - 1 \) of the variables \( x_1, x_2, \ldots, x_m \) say, all but \( x_j \), then \( h_k \) is the composition of a function from \( \mathbb{R} \) to \( \mathbb{R}^q \) with a function from \( \mathbb{R}^q \) to \( \mathbb{R} \). Thus we may use the chain rule from Section 3.3 to compute \( \frac{\partial}{\partial x_j} h_k(c) \), namely,

\[
    \frac{\partial}{\partial x_j} h_k(c) = \nabla f_k(g(c)) \cdot \left( \frac{\partial}{\partial x_j} g_1(c), \frac{\partial}{\partial x_j} g_2(c), \ldots, \frac{\partial}{\partial x_j} g_q(c) \right) \\
    = \frac{\partial}{\partial x_1} f_k(g(c)) \frac{\partial}{\partial x_j} g_1(c) + \frac{\partial}{\partial x_2} f_k(g(c)) \frac{\partial}{\partial x_j} g_2(c) + \cdots + \frac{\partial}{\partial x_q} f_k(g(c)) \frac{\partial}{\partial x_j} g_q(c). \tag{4.2.13}
\]

Hence \( \frac{\partial}{\partial x_j} h_k(c) \) is equal to the dot product of the \( k \)th row of \( Df(g(c)) \) with the \( j \)th column of \( Dg(c) \). Moreover, if \( g \) is \( C^1 \) on an open ball about \( c \) and \( f \) is \( C^1 \) on an open ball about \( g(c) \), then (4.2.13) shows that \( \frac{\partial}{\partial x_j} h_k \) is continuous on an open ball about \( c \). It follows from our results in Section 3.3 that \( h \) is differentiable at \( c \). Since \( \frac{\partial}{\partial x_j} h_k \) is the entry in the \( k \)th row and \( j \)th column of \( Dh(c) \), (4.2.13) implies \( Dh(c) = Df(g(c))Dg(c) \). This result, the chain rule, may be proven without assuming that \( f \) and \( g \) are both \( C^1 \), and so we state the more general result in the following theorem.

Figure 4.2.1 Torus with a tangent plane
Chain Rule  If \( g : \mathbb{R}^m \to \mathbb{R}^q \) is differentiable at \( c \) and \( f : \mathbb{R}^q \to \mathbb{R}^n \) is differentiable at \( g(c) \), then \( f \circ g \) is differentiable at \( c \) and
\[
D(f \circ g)(c) = Df(g(c))Dg(c).
\] (4.2.14)

Equivalently, the chain rule says that if \( A \) is the best affine approximation to \( g \) at \( c \) and \( B \) is the best affine approximation to \( f \) at \( g(c) \), then \( B \circ A \) is the best affine approximation to \( f \circ g \) at \( c \). That is, the best affine approximation to a composition of functions is the composition of the individual best affine approximations.

Example  Suppose \( g : \mathbb{R}^2 \to \mathbb{R}^3 \) is defined by
\[
g(s, t) = (\cos(s) \sin(t), \sin(s) \sin(t), \cos(t))
\]
and \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) is defined by
\[
f(x, y, z) = (10xyz, x^2 - yz).
\]

Then
\[
Dg(s, t) = \begin{bmatrix}
-\sin(s) \sin(t) & \cos(s) \cos(t) \\
\cos(s) \sin(t) & \sin(s) \cos(t) \\
0 & -\sin(t)
\end{bmatrix}
\]
and
\[
Df(x, y, z) = \begin{bmatrix}
10yz & 10xz & 10xy \\
2x & -z & -y
\end{bmatrix}.
\]

Let \( h(s, t) = f(g(s, t)) \). To find \( Dh \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \), we first note that
\[
g \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right),
\]

\[
Dg \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = \begin{bmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
0 & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\]
and
\[
Df \left( g \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \right) = Df \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right) = \begin{bmatrix}
\frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & \frac{5}{2} \\
\sqrt{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2}
\end{bmatrix}.
\]
Thus

\[
Dh \left( \frac{\pi}{4}, \frac{\pi}{4} \right) = Df \left( g \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \right) Dg \left( \frac{\pi}{4}, \frac{\pi}{4} \right)
\]

\[
= \begin{bmatrix}
\frac{5}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \\
\frac{2}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
0 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & \frac{5}{2\sqrt{2}} \\
-\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{1}{2}
\end{bmatrix}
\]

Problems

1. Find the best affine approximation for each of the following functions at the specified point \( c \).
   (a) \( f(x, y) = (x^2 + y^2, 3xy), \ c = (1, 2) \)
   (b) \( g(x, y, z) = (\sin(x + y + z), xy \cos(z)), \ c = (0, \frac{\pi}{4}, \frac{\pi}{4}) \)
   (c) \( h(s, t) = (3s^2 + t, s - t, 4st^2, 4t - s), \ c = (-1, 3) \)

2. Each of the following functions parametrizes a surface \( S \) in \( \mathbb{R}^3 \). In each case, find parametric equations for the tangent plane \( P \) passing through the point \( f(s_0, t_0) \). Plot \( S \) and \( P \) together.
   (a) \( f(s, t) = (t \cos(s), t \sin(s), t), \ (s_0, t_0) = \left( \frac{\pi}{2}, 2 \right) \)
   (b) \( f(s, t) = (t^2 \cos(s), t^2, t^2 \sin(s)), \ (s_0, t_0) = (0, 1) \)
   (c) \( f(s, t) = (\cos(s) \sin(t), \sin(s) \sin(t), \cos(t)), \ (s_0, t_0) = \left( \frac{\pi}{2}, \frac{\pi}{4} \right) \)
   (d) \( f(s, t) = (3 \cos(s) \sin(t), \sin(s) \sin(t), 2 \cos(t)), \ (s_0, t_0) = \left( \frac{\pi}{4}, \frac{\pi}{4} \right) \)
   (e) \( f(s, t) = ((4 + 2 \cos(t)) \cos(s), (4 + 2 \cos(t)) \sin(s), 2 \sin(t)), \ (s_0, t_0) = \left( \frac{3\pi}{4}, \frac{\pi}{4} \right) \)

3. Let \( S \) be the graph of a function \( f : \mathbb{R}^2 \to \mathbb{R} \). Define the function \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) by \( F(s, t) = (s, t, f(s, t)) \). We may find an equation for the plane tangent to \( S \) at \( (s_0, t_0, f(s_0, t_0)) \) using either the techniques of Section 3.3 (looking at \( S \) as the graph of \( f \)) or the techniques of this section (looking at \( S \) as a surface parametrized by \( F \)). Verify that these two approaches yield equations for the same plane, both in the special case when \( f(s, t) = s^2 + t^2 \) and \( (s_0, t_0) = (1, 2) \), and in the general case.

4. Use the chain rule to find the derivative of \( f \circ g \) at the point \( c \) for each of the following.
   (a) \( f(x, y) = (x^2 + y^2, x - y), \ g(s, t) = (3st, s^2 - 4t), \ c = (1, -2) \)
   (b) \( f(x, y, z) = (4xy, 3xz), \ g(s, t) = \left( st^2 - 4t, s^2, \frac{4}{st} \right), \ c = (-2, 3) \)
   (c) \( f(x, y) = (3x + 4y, 2x^2y, x - y), \ g(s, t, u) = (4s - 3t + u, 5st^2), \ c = (1, -2, 3) \)
5. Suppose 
\[ x = f(u, v), \]
\[ y = g(u, v), \]
and 
\[ u = h(s, t), \]
\[ v = k(s, t). \]

(a) Show that 
\[ \frac{\partial x}{\partial s} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial s} \]
and 
\[ \frac{\partial x}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial t}. \]

(b) Find similar expressions for \[ \frac{\partial y}{\partial s} \] and \[ \frac{\partial y}{\partial t}. \]

6. Use your results in Problem 5 to find \[ \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial s}, \text{ and } \frac{\partial y}{\partial t} \]
when 
\[ x = u^2 v, \]
\[ y = 3u - v, \]
and 
\[ u = 4t^2 - s^2, \]
\[ v = \frac{4t}{s}. \]

7. Suppose \( T \) is a function of \( x \) and \( y \) where 
\[ x = r \cos(\theta), \]
\[ y = r \sin(\theta). \]

Show that 
\[ \frac{\partial T}{\partial r} = \frac{\partial T}{\partial x} \cos(\theta) + \frac{\partial T}{\partial y} \sin(\theta) \]
and 
\[ \frac{\partial T}{\partial \theta} = -r \frac{\partial T}{\partial x} \sin(\theta) + r \frac{\partial T}{\partial y} \cos(\theta). \]

8. Suppose the temperature at a point \((x, y)\) in the plane is given by 
\[ T = 100 - \frac{20}{\sqrt{1 + x^2 + y^2}}. \]

(a) If \((r, \theta)\) represents the polar coordinates of \((x, y)\), use Problem 7 to find \[ \frac{\partial T}{\partial r} \] and \[ \frac{\partial T}{\partial \theta} \] when \( r = 4 \) and \( \theta = \frac{\pi}{6}. \)

(b) Show that \[ \frac{\partial T}{\partial \theta} = 0 \] for all values of \( r \) and \( \theta \). Can you explain this result geometrically?
9. Let $T$ be the torus parametrized by

$$
\begin{align*}
  x &= (4 + 2 \cos(t)) \cos(s), \\ 
  y &= (4 + 2 \cos(t)) \sin(s), \\ 
  z &= 2 \sin(t),
\end{align*}
$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq 2\pi$.

(a) If $U$ is a function of $x$, $y$, and $z$, find general expressions for $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$.

(b) Suppose

$$
U = 80 - 40e^{-\frac{1}{20}(x^2+y^2+z^2)}
$$

gives the temperature at a point $(x, y, z)$ on $T$. Find expressions for $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$ in this case. What is the geometrical interpretation of these quantities?

(c) Evaluate $\frac{\partial U}{\partial s}$ and $\frac{\partial U}{\partial t}$ in the particular case $s = \frac{\pi}{4}$ and $t = \frac{\pi}{4}$. 