Now that we have a basic understanding of the geometry of $\mathbb{R}^n$, we are in a position to start the study of calculus of more than one variable. We will break our study into three pieces. In this chapter we will consider functions $f : \mathbb{R} \to \mathbb{R}^n$, in Chapter 3 we will study functions $f : \mathbb{R}^n \to \mathbb{R}$, and finally in Chapter 4 we will consider the general case of functions $f : \mathbb{R}^m \to \mathbb{R}^n$.

**Parametrizations of curves**

We begin with some terminology and notation. Given a function $f : \mathbb{R} \to \mathbb{R}^n$, let

$$f_k(t) = k\text{th coordinate of } f(t)$$

for $k = 1, 2, \ldots, n$. We call $f_k : \mathbb{R} \to \mathbb{R}$ the $k\text{th coordinate function}$ of $f$. Note that $f_k$ has the same domain as $f$ and that, for any point $t$ in the domain of $f$,

$$f(t) = (f_1(t), f_2(t), \ldots, f_n(t)).$$

(2.1.2)

If the domain of $f$ is an interval $I$, then the range of $f$, that is, the set

$$C = \{x : x = f(t) \text{ for some } t \text{ in } I\},$$

(2.1.3)

is called a curve with parametrization $f$. The equation $x = f(t)$, where $x$ is in $\mathbb{R}^n$, is a vector equation for $C$ and, writing $x = (x_1, x_2, \ldots, x_n)$, the equations

$$x_1 = f_1(t),$$

$$x_2 = f_2(t),$$

$$\vdots$$

$$x_n = f_n(t),$$

(2.1.4)

are parametric equations for $C$.

**Example** Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (\cos(t), \sin(t))$$

for $0 \leq t \leq 2\pi$. Then for every value of $t$, $f(t)$ is a point on the circle $C$ of radius 1 with center at $(0,0)$. Note that $f(0) = (1,0)$, $f\left(\frac{\pi}{2}\right) = (0,1)$, $f(\pi) = (-1,0)$, $f\left(\frac{3\pi}{2}\right) = (0,-1)$,
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Figure 2.1.1 $f(t) = (\cos(t), \sin(t))$

and $f(2\pi) = (1,0) = f(0)$. In fact, as $t$ goes from 0 to $2\pi$, $f(t)$ traverses $C$ exactly once in the counterclockwise direction. Thus $f$ is a parametrization of the unit circle $C$. If we denote a point in $\mathbb{R}^2$ by $(x,y)$, then

$$x = \cos(t),$$
$$y = \sin(t),$$

are parametric equations for $C$. See Figure 2.1.1. The coordinate functions are

$$f_1(t) = \cos(t),$$
$$f_2(t) = \sin(t),$$

although we frequently write these as simply

$$x(t) = \cos(t),$$
$$y(t) = \sin(t).$$

Example Consider $g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t) = (\sin(2\pi t), \cos(2\pi t))$$

for $0 \leq t \leq 2$. Then $g$ also parametrizes the unit circle $C$ centered at the origin, the same as $f$ in the previous example. However, there is a difference: $g(0) = (0,1), g\left(\frac{1}{4}\right) = (1,0), \ldots.$
$g\left(\frac{1}{2}\right) = (0, -1), \ g\left(\frac{3}{4}\right) = (-1, 0), \text{ and } g(1) = (0, 1) = g(0)$, at which point $g$ starts to repeat its values. Hence $g(t)$, starting at $(0, 1)$, traverses $C$ twice in the clockwise direction as $t$ goes from 0 to 2.

**Example** More generally, suppose $a$, $b$, and $\alpha$ are real numbers, with $a > 0$, $b > 0$, and $\alpha \neq 0$, and let

$$x(t) = a \cos(\alpha t), \quad y(t) = b \sin(\alpha t).$$

Then

$$\frac{(x(t))^2}{a^2} + \frac{(y(t))^2}{b^2} = \cos^2(\alpha t) + \sin^2(\alpha t) = 1,$$

so $(x(t), y(t))$ is a point on the ellipse $E$ with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

shown in Figure 2.1.2. Thus the function

$$f(t) = (a \cos(\alpha t), b \cos(\alpha t))$$

parametrizes the ellipse $E$, traversing the complete ellipse as $t$ goes from 0 to $\left|\frac{2\pi}{\alpha}\right|$.

**Example** Define $f : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$f(t) = (t \cos(t), t \sin(t))$$

for $-\infty < t < \infty$. Then for negative values of $t$, $f(t)$ spirals into the origin as $t$ increases, while for positive values of $t$, $f(t)$ spirals away from the origin. Part of this curve parametrized by $f$ is shown in Figure 2.1.3.
Example  Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (3 - 4t, 2 + 3t)$$

for $-\infty < t < \infty$. Then

$$f(t) = t(-4, 3) + (3, 2),$$

so $f$ is a parametrization of the line through the point $(3, 2)$ in the direction of $(-4, 3)$.

In general, a function $f : \mathbb{R} \to \mathbb{R}^n$ defined by $f(t) = tv + p$, where $v \neq 0$ and $p$ are vectors in $\mathbb{R}^n$, parametrizes a line in $\mathbb{R}^n$.

Example  Suppose $g : \mathbb{R} \to \mathbb{R}^3$ is defined by

$$g(t) = (4 \cos(t), 4 \sin(t), t)$$

for $-\infty < t < \infty$. If we denote the coordinate functions by

$$x(t) = 4 \cos(t),$$
$$y(t) = 4 \sin(t),$$
$$z(t) = t,$$

then

$$(x(t))^2 + (y(t))^2 = 16 \cos^2(t) + 16 \sin^2(t) = 16.$$  

Hence $g(t)$ always lies on a cylinder of radius 1 centered about the $z$-axis. As $t$ increases, $g(t)$ rises steadily as it winds around this cylinder, completing one trip around the cylinder.
Figure 2.1.4 The helix \( f(t) = (4 \cos(t), 4 \sin(t), t), \; -2\pi \leq t \leq 2\pi \)

over every interval of length \( 2\pi \). In other words, \( g \) parametrizes a helix, part of which is shown in Figure 2.1.4.

**Limits in \( \mathbb{R}^n \)**

As was the case in one-variable calculus, limits are fundamental for understanding ideas such as continuity and differentiability. We begin with the definition of the limit of a sequence of points in \( \mathbb{R}^m \).

**Definition** Let \( \{x_n\} \) be a sequence of points in \( \mathbb{R}^m \). We say that the limit of \( \{x_n\} \) as \( n \) approaches infinity is \( a \), written \( \lim_{n \to \infty} x_n = a \), if for every \( \epsilon > 0 \) there is a positive integer \( N \) such that

\[
\|x_n - a\| < \epsilon \quad (2.1.5)
\]

whenever \( n > N \).

Notice that this definition involves only a slight modification of the definition for the limit of a sequence of real numbers, namely, the use of the norm of a vector instead of the
absolute value of a real number in (2.1.5). In words, \( \lim_{n \to \infty} x_n = a \) if, given any \( \epsilon > 0 \), we can always find a point in the sequence beyond which all terms of the sequence lie within \( B^n(a, \epsilon) \), the open ball of radius \( \epsilon \) centered at \( a \).

**Example** Suppose

\[
x_n = \left( 1 - \frac{1}{n}, \frac{2}{n} \right)
\]

for \( n = 1, 2, 3, \ldots \). Since

\[
\lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) = 1
\]

and

\[
\lim_{n \to \infty} \frac{2}{n} = 0,
\]

we should have

\[
\lim_{n \to \infty} x_n = (1, 0).
\]

To verify this, we first note that

\[
\|x_n - (1, 0)\| = \left\| \left( -\frac{1}{n}, \frac{2}{n} \right) \right\| = \sqrt{\frac{1}{n^2} + \frac{4}{n^2}} = \frac{\sqrt{5}}{n}.
\]

Hence \( \|x_n - (1, 0)\| < \epsilon \) whenever \( n > \frac{\sqrt{5}}{\epsilon} \). That is, if we let \( N \) be any integer greater than or equal to \( \frac{\sqrt{5}}{\epsilon} \), then \( \|x_n - (1, 0)\| < \epsilon \) whenever \( n > N \), verifying that

\[
\lim_{n \to \infty} x_n = (1, 0).
\]

See Figure 2.1.5.

Put another way, the definition of the limit of a sequence in \( \mathbb{R}^m \) says that a sequence \( \{x_n\} \) in \( \mathbb{R}^m \) converges to \( a \) in \( \mathbb{R}^m \) if and only if the sequence of real numbers \( \{\|x_n - a\|\} \)
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converges to 0. That is, \( \lim_{n \to \infty} x_n = a \) if and only if \( \lim_{n \to \infty} \|x_n - a\| = 0 \). Moreover, if we let \( x_n = (x_{n1}, x_{n2}, \ldots, x_{nm}) \) and \( a = (a_1, a_2, \ldots, a_m) \), then

\[
\|x_n - a\| = \sqrt{(x_{n1} - a_1)^2 + (x_{n2} - a_2)^2 + \cdots + (x_{nm} - a_m)^2},
\]

so \( \lim_{n \to \infty} \|x_n - a\| = 0 \) if and only if

\[
\lim_{n \to \infty} \sqrt{(x_{n1} - a_1)^2 + (x_{n2} - a_2)^2 + \cdots + (x_{nm} - a_m)^2} = 0.
\]

But (2.1.7) can occur only when \( \lim_{n \to \infty} (x_{nk} - a_k)^2 = 0 \) for \( k = 1, 2, \ldots, m \). Hence \( \lim_{n \to \infty} x_n = a \) if and only if \( \lim_{n \to \infty} x_{nk} = a_k \) for \( k = 1, 2, \ldots, m \).

**Proposition** Suppose \( \{x_n\} \) is a sequence in \( \mathbb{R}^m \), \( x_n = (x_{n1}, x_{n2}, \ldots, x_{nm}) \), and \( a = (a_1, a_2, \ldots, a_m) \). Then \( \lim_{n \to \infty} x_n = a \) if and only if \( \lim_{n \to \infty} x_{nk} = a_k \) for \( k = 1, 2, \ldots, m \).

This proposition tells us that to compute the limit of a sequence in \( \mathbb{R}^m \), we need only compute the limit of each coordinate separately, thus reducing the problem of computing limits in \( \mathbb{R}^m \) to the problem of finding limits of sequences of real numbers.

**Example** If

\[
x_n = \left( \frac{2 - n}{n^2}, \sin \left( \frac{1}{n} \right), \cos \left( \frac{3}{n} \right) \right),
\]

\( n = 1, 2, 3, \ldots \), then

\[
\lim_{n \to \infty} x_n = \left( \lim_{n \to \infty} \frac{2 - n}{n^2}, \lim_{n \to \infty} \sin \left( \frac{1}{n} \right), \lim_{n \to \infty} \cos \left( \frac{3}{n} \right) \right) = (0, 0, 1).
\]

We may now define the limit of a function \( f : \mathbb{R} \to \mathbb{R}^m \) at a real number \( c \). Notice that the definition is identical to the definition of a limit for a real-valued function \( f : \mathbb{R} \to \mathbb{R} \).

**Definition** Let \( c \) be a real number, let \( I \) be an open interval containing \( c \), and let \( J = \{ t : t \text{ is in } I, t \neq c \} \). Suppose \( f : \mathbb{R} \to \mathbb{R}^m \) is defined for all \( t \) in \( J \). Then we say that the limit of \( f(t) \) as \( t \) approaches \( c \) is \( a \), denoted \( \lim_{t \to c} f(t) = a \), if for every sequence of real numbers \( \{t_n\} \) in \( J \),

\[
\lim_{n \to \infty} f(t_n) = a
\]

whenever \( \lim_{n \to \infty} t_n = c \).

As in one-variable calculus, we may define the limit of \( f(t) \) as \( t \) approaches \( c \) from the right, denoted

\[
\lim_{t \to c^+} f(t),
\]

by restricting to sequences \( \{t_n\} \) with \( t_n > c \) for \( n = 1, 2, 3, \ldots \), and the limit of \( f(t) \) as \( t \) approaches \( c \) from the left, denoted

\[
\lim_{t \to c^-} f(t),
\]
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by restricting to sequences \( \{t_n\} \) with \( t_n < c \) for \( n = 1, 2, 3, \ldots \). Moreover, the following useful proposition follows immediately from our definition and the previous proposition.

**Proposition** Suppose \( f : \mathbb{R} \rightarrow \mathbb{R}^m \) with

\[
f(t) = (f_1(t), f_2(t), \ldots, f_m(t)).
\]

The for any real number \( c \),

\[
\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t)). \quad (2.1.9)
\]

Hence the problem of computing limits for functions \( f : \mathbb{R} \rightarrow \mathbb{R}^m \) reduces to the problem of computing limits of the coordinate functions \( f_k : \mathbb{R} \rightarrow \mathbb{R}, \ k = 1, 2, \ldots, m, \) a familiar problem from one-variable calculus. The analogous statements for limits from the right and left also hold.

**Example** If \( f(t) = (t^2 - 1, \sin(t), \cos(t)) \) is a function from \( \mathbb{R} \) to \( \mathbb{R}^3 \), then, for example,

\[
\lim_{t \to \pi} f(t) = \left( \lim_{t \to \pi} (t^2 - 1), \lim_{t \to \pi} \sin(t), \lim_{t \to \pi} \cos(t) \right) = (\pi^2 - 1, 0, -1).
\]

Definitions for continuity also follow the pattern of the related definitions in one-variable calculus.

**Definition** Suppose \( f : \mathbb{R} \rightarrow \mathbb{R}^m \). We say \( f \) is **continuous at a point** \( c \) if

\[
\lim_{t \to c} f(t) = f(c). \quad (2.1.10)
\]

We say \( f \) is **continuous from the right at** \( c \) if

\[
\lim_{t \to c^+} f(t) = f(c) \quad (2.1.11)
\]

and **continuous from the left at** \( c \) if

\[
\lim_{t \to c^-} f(t) = f(c). \quad (2.1.12)
\]

We say \( f \) is **continuous** on an open interval \((a, b)\) if \( f \) is continuous at every point \( c \) in \((a, b)\) and we say \( f \) is **continuous** on a closed interval \([a, b]\) if \( f \) is continuous on the open interval \((a, b)\), continuous from the right at \( a \), and continuous from the left at \( b \).

If \( f(t) = (f_1(t), f_2(t), \ldots, f_m(t)) \), then \( f \) is continuous at a point \( c \) if and only if

\[
\lim_{t \to c} f(t) = (\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \ldots, \lim_{t \to c} f_m(t)) = (f_1(c), f_2(c), \ldots, f_m(c)),
\]

which is true if and only if \( \lim_{t \to c} f_k(t) = f_k(c) \) for \( k = 1, 2, \ldots, m \). In other words, we have the following useful proposition.
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**Proposition** A function \( f : \mathbb{R} \to \mathbb{R}^m \) with \( f(t) = (f_1(t), f_2(t), \ldots, f_m(t)) \) is continuous at a point \( c \) if and only if the coordinate functions \( f_1, f_2, \ldots, f_m \) are each continuous at \( c \).

Similar statements hold for continuity from the right and from the left.

**Example** The function \( f : \mathbb{R} \to \mathbb{R}^3 \) defined by

\[
f(t) = (\sin(t^2), t^3 + 4, \cos(t))
\]

is continuous on the interval \((-\infty, \infty)\) since each of its coordinate functions is continuous on \((-\infty, \infty)\).

**Problems**

1. Plot the curves parametrized by the following functions over the specified intervals \( I \).
   (a) \( f(t) = (3t + 1, 2t - 1), \ I = [-5, 5] \)
   (b) \( g(t) = (t, t^2), \ I = [-3, 3] \)
   (c) \( f(t) = (3 \cos(t), 3 \sin(t)), \ I = [0, 2\pi] \)
   (d) \( h(t) = (3 \cos(t), 3 \sin(t)), \ I = [0, \pi] \)
   (e) \( f(t) = (4 \cos(2t), 2 \sin(2t)), \ I = [0, \pi] \)
   (f) \( g(t) = (-4 \cos(t), 2 \sin(t)), \ I = [0, \pi] \)
   (g) \( h(t) = (t \sin(3t), t \cos(3t)), \ I = [-\pi, \pi] \)

2. Plot the curves parametrized by the following functions over the specified intervals \( I \).
   (a) \( f(t) = (t + 1, 2t - 1, 3t), \ I = [-4, 4] \)
   (b) \( g(t) = (\cos(t), t, \sin(t)), \ I = [0, 4\pi] \)
   (c) \( f(t) = (t \cos(2t), t \sin(2t), t), \ I = [-10, 10] \)
   (d) \( h(t) = (\cos(2t), \sin(2t), \sqrt{3}), \ I = [0, 9] \)

3. Plot the curves parametrized by the following functions over the specified intervals \( I \).
   (a) \( f(t) = (\cos(4\pi t), \sin(5\pi t)), \ I = [-0.5, 0.5] \)
   (b) \( f(t) = (\cos(6\pi t), \sin(7\pi t)), \ I = [-0.5, 0.5] \)
   (c) \( h(t) = (\cos^3(t), \sin^3(t)), \ I = [0, 2\pi] \)
   (d) \( g(t) = (\cos(2\pi t), \sin(2\pi t), \sin(4\pi t)), \ I = [0, 1] \)
   (e) \( f(t) = (\sin(4t) \cos(t), \sin(4t) \sin(t)), \ I = [0, 2\pi] \)
   (f) \( h(t) = ((1 + 2 \cos(t)) \cos(t), (1 + 2 \cos(t)) \sin(t)), \ I = [0, 2\pi] \)

4. Suppose \( g : \mathbb{R} \to \mathbb{R} \) and we define \( f : \mathbb{R} \to \mathbb{R}^2 \) by \( f(t) = (t, g(t)) \). Describe the curve parametrized by \( f \).

5. For each of the following, compute \( \lim_{n \to \infty} x_n \).
   (a) \( x_n = \left( \frac{n + 1}{2n + 3}, 3 - \frac{1}{n} \right) \)
   (b) \( x_n = \left( \sin \left( \frac{n - 1}{n} \right), \cos \left( \frac{n - 1}{n} \right), \frac{n - 1}{n} \right) \)
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(c) \( x_n = \left( \frac{2n - 1}{n^2 + 1}, \frac{3n + 4}{n + 1}, 4 - \frac{6n + 1}{n^2}, \frac{6n + 1}{2n^2 + 5} \right) \)

6. Let \( f : \mathbb{R} \to \mathbb{R}^3 \) be defined by

\[
f(t) = \left( \frac{\sin(t)}{t}, \cos(t), 3t^2 \right).
\]

Evaluate the following.

(a) \( \lim_{t \to \pi} f(t) \)

(b) \( \lim_{t \to 1} f(t) \)

(c) \( \lim_{t \to 0} f(t) \)

7. Discuss the continuity of each of the following functions.

(a) \( f(t) = (t^2 + 1, \cos(2t), \sin(3t)) \)

(b) \( g(t) = (\sqrt{t + 1}, \tan(t)) \)

(c) \( f(t) = \left( \frac{1}{t^2 - 1}, \sqrt{1 - t^2}, \frac{1}{t} \right) \)

(d) \( g(t) = (\cos(4t), 1 - \sqrt{3t + 1}, \sin(5t), \sec(t)) \)

8. Let \( f : \mathbb{R} \to \mathbb{R}^3 \) be defined by \( f(t) = (t^2, 3t, 2t + 1) \). Find

\[
\lim_{h \to 0} \frac{f(t + h) - f(t)}{h}.
\]