7.1 Notation and terminology

If \( S \) is a set of complex numbers and \( f : S \to \mathbb{C} \) is defined for all \( z \in S \), we call \( S \) the domain of \( f \). Note that the domain of a function \( f \) need not be a domain. As is customary, when a domain is not specified for a function \( f \), it is assumed to be the largest possible set for which \( f \) is defined.

Example 7.1. The domain of

\[
f(z) = \frac{1}{z^2 + 1}
\]

is

\[
S = \{ z \in \mathbb{C} : z \neq \pm i \}.
\]

If \( f : S \to \mathbb{C} \), then \( f(z) \) is a complex number and so we may write

\[
f(z) = u + iv
\]

for real numbers \( u \) and \( v \) which depend on \( z \). Now \( z = x + iy \) for some real numbers \( x \) and \( y \), so in fact \( u \) and \( v \) are functions of real numbers \( x \) and \( y \). That is, we may write

\[
f(z) = f(x + iy) = u(x, y) + iv(x, y),
\]

where \( u \) and \( v \) are real-valued functions of two real variables. If instead we write \( z \) in polar coordinates, say, \( z = re^{i\theta} \), then we may write

\[
f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta).
\]
Example 7.2. If \( f(z) = z^3 \), then
\[
f(x + iy) = (x + iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3),
\]
so
\[
u(x, y) = x^3 - 3xy^2
\]
and
\[
v(x, y) = 3x^2y - y^3.
\]
In polar coordinates,
\[
f(re^{i\theta}) = r^3(\cos(3\theta) + i\sin(3\theta)),
\]
so
\[
u(r, \theta) = r^3 \cos(3\theta)
\]
and
\[
v(r, \theta) = r^3 \sin(3\theta).
\]

Definition 7.1. Given complex numbers \( a_0, a_1, \ldots, a_n \), with \( a_n \neq 0 \), we call
\[
P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n
\]
a polynomial of degree \( n \). If \( P \) and \( Q \) are polynomials, we call
\[
R(z) = \frac{P(z)}{Q(z)}
\]
a rational function.

In some instances we will need to consider a relationship which associates multiple values with a given input value, as illustrated in the following example.

Example 7.3. Recall that if \( z = re^{i\theta} \in \mathbb{C} \), \( z \neq 0 \) and \(-\pi < \theta \leq \pi\), then the two square roots of \( z \) are
\[
c_0 = \sqrt{re^{i\theta}}
\]
and
\[
c_1 = \sqrt{re^{i(\theta + \pi)}}
\]
Figure 7.1: Plot of the imaginary part of the principal square root of $z$

\[ z^{\frac{1}{2}} = \pm \sqrt{re^{i\theta}}. \]

We say that $z^{\frac{1}{2}}$ is a *multi-valued function*. We may create a function (that is, a *single-valued* function) $f : \mathbb{C} \rightarrow \mathbb{C}$ from this multi-valued function by defining $f(0) = 0$ and

\[ f(z) = \sqrt{re^{i\theta}} \]

for $z = re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. The plot in Figure 7.1 displays the imaginary part of $f(z)$, the blue curve illustrating the curve on the surface above the unit circle. Note the split in the surface along the negative real axis.