Lecture 5:
Roots of Complex Numbers

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5.1 Roots
Suppose \( z_0 \) is a complex number and, for some positive integer \( n \), \( z \) is an \( n \)th root of \( z_0 \); that is, \( z^n = z_0 \). Now if \( z = re^{i\theta} \) and \( z_0 = r_0e^{i\theta_0} \), then we must have
\[
r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi
\]
for some integer \( k \). Hence we must have
\[
r = \sqrt[n]{r_0},
\]
where \( \sqrt[n]{r_0} \) denotes the unique positive \( n \)th root of the real number \( r \), and
\[
\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}
\]
for some integer \( k \). Hence the numbers
\[
z = \sqrt[n]{r_0} \exp \left( i \left( \frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right), \quad \text{for } k = 0, \pm 1, \pm 2, \ldots,
\]
are all \( n \)th roots of \( z_0 \), with
\[
z = \sqrt[n]{r_0} \exp \left( i \left( \frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right), \quad \text{for } k = 0, 1, 2, \ldots, n - 1,
\]
being the \( n \) distinct \( n \)th roots of \( z_0 \). We let \( z_0^{\frac{1}{n}} \) denote this set of \( n \)th roots of \( z_0 \). Note that these roots are equally spaced around the circle of radius \( \sqrt[n]{r_0} \). If \( \theta_0 = \text{Arg} \ z_0 \), we call

\[
z = \sqrt[n]{r_0} e^{i \frac{\theta_0}{n}},
\]

the principal root of \( z_0 \).

### 5.2 Roots of unity

**Example 5.1.** Since

\[
1 = 1 e^{i(0 + 2k\pi)} \text{ for } k = 0, \pm 1, \pm 2, \ldots,
\]

for any positive integer \( n \),

\[
z = e^{i \frac{2k\pi}{n}}
\]

is an \( n \)th root of 1 for any integer \( k \), which we call an \( n \)th root of unity. If we let

\[
\omega_n = e^{i \frac{2\pi}{n}},
\]

then, for any integer \( k \),

\[
\omega_n^k = e^{i \frac{2k\pi}{n}}
\]

Hence the distinct \( n \)th roots of unity are

\[
1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.
\]

For example, when \( n = 2 \),

\[
\omega_2 = e^{i\pi} = -1,
\]

and the roots of unity are simply 1 and \(-1\). When \( n = 3 \),

\[
w_3 = e^{i \frac{2\pi}{3}},
\]

and the distinct third roots of unity are

\[
1, e^{i \frac{2\pi}{3}}, e^{i \frac{4\pi}{3}};
\]

that is

\[
1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i.
\]
When $n = 4$, 
\[ \omega_4 = e^{i \frac{2\pi}{4}} = e^{i \frac{\pi}{2}} = i, \]
and the distinct roots fourth roots of unity are 
\[1, i, -1, -i.\]

Note that the $n$th roots of unity are equally spaced around the unit circle. For example, the third roots of unity form an equilateral triangle in the unit circle, as shown in Figure 5.1.

More generally, if $c$ is any particular $n$th root of $z_0$, then the distinct $n$th roots of $z_0$ are 
\[c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}.\]

**Example 5.2.** To find the cube roots of $-27i$, we first note that 
\[-27i = 27 \exp \left( i \left( -\frac{\pi}{2} + 2k\pi \right) \right), \text{ for } k = 0, \pm 1, \pm 2, \ldots.\]
Figure 5.2: The cube roots of $-27i$ form an equilateral triangle

Hence the principal cube root of $-27i$ is

$$c_0 = 3e^{-i\frac{\pi}{3}} = 3 \left( \cos \left( \frac{\pi}{6} \right) - i \sin \left( \frac{\pi}{6} \right) \right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i.$$ 

The remaining cube roots are

$$c_1 = 3e^{i\frac{\pi}{6}} = 3 \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) = 3i,$$

and

$$c_2 = 3e^{i\frac{7\pi}{6}} = 3 \left( \cos \left( \frac{7\pi}{6} \right) + i \sin \left( \frac{7\pi}{6} \right) \right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i,$$

which are simply $c_0\omega_3$ and $c_0\omega_3^2$, where

$$\omega_3 = e^{i\frac{2\pi}{3}}.$$

4