40.1 Continuity

Theorem 40.1. Suppose the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)$$

has radius of convergence $R$ and let $D = \{ z \in \mathbb{C} : |z - z_0| < R \}$. If, for $z \in D$, we let

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

then $S$ is continuous on $D$.

Proof. Let $z \in D$. We need to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(w) - S(z)| < \epsilon$$

whenever $|w - z| < \delta$. Choose a positive real number $R_0$ such that $|z - z_0| < R_0 < R$ and, for any positive integer $N$, let

$$S_N(w) = \sum_{n=0}^{N-1} a_n(w - z_0)^n$$
and
\[ \rho_N(w) = \sum_{n=N}^{\infty} a_n(w - z_0)^n. \]

Since the power series converges uniformly on the closed disk \(|w - z_0| \leq R_0|\), we may choose a positive integer \(N_\epsilon\) such that
\[ |\rho_{N_\epsilon}(w)| < \frac{\epsilon}{3}, \]
for all \(w\) with \(|w - z_0| \leq R_0\). Since \(S_{N_\epsilon}(z)\) is continuous (since it is a polynomial), we may choose a \(\delta_1 > 0\) such that
\[ |S_{N_\epsilon}(w) - S_{N_\epsilon}(z)| < \frac{\epsilon}{3} \]
whenever \(|w - z| < \delta_1\). Let \(\delta\) be the smaller of \(\delta_1\) and \(R_0 - |z - z_0|\). Then, for all \(w\) with \(|w - z| < \delta\), we have
\[
|S(w) - S(z)| = |(S_{N_\epsilon}(w) + \rho_{N_\epsilon}(w)) - (S_{N_\epsilon}(z) + \rho_{N_\epsilon}(z))|
\leq |S_{N_\epsilon}(w) - S_{N_\epsilon}(z)| + |\rho_{N_\epsilon}(w)| + |\rho_{N_\epsilon}(z)|
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon.
\]

\[ \square \]

### 40.2 Series with negative powers

Now suppose a series of the form
\[ \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \]
converges at a point \(z_1 \neq z_0\). Let \(w_1 = \frac{1}{z_1 - z_0}\). Then the series
\[ \sum_{n=1}^{\infty} b_n w_1^n \]
converges, and so the power series
\[ \sum_{n=1}^{\infty} b_n w^n \]

converges absolutely for all \( w \) with \( |w| < |w_1| \). If we let \( w = \frac{1}{z-z_0} \), this says that

\[
\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}
\]

converges absolutely whenever

\[
\frac{1}{|z-z_0|} < \frac{1}{|z_1-z_0|},
\]

that is, whenever \( |z-z_0| > R_1 \), where \( R_1 = |z_1-z_0| \). Hence

\[
\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}
\]

converges absolutely for all \( z \) in the exterior of the circle \( |z-z_0| = R_1 \).

Moreover, the function to which the series converges is continuous.

More generally, one may show that if the series

\[
f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}
\]

converges on an annulus \( R_1 < |z-z_0| < R_2 \), then, for any \( R_1 < \rho_1 < \rho_2 < R_2 \), both series converge uniformly on the closed annulus \( \rho_1 \leq |z-z_0| \leq \rho_2 \).