4.1 Polar coordinates

Recall: If \((x, y)\) is a point in the plane, \((x, y) \neq (0, 0)\), \(r\) is the distance from \((x, y)\) to the origin, and \(\theta\) is the angle between the \(x\)-axis and the line passing through \((x, y)\) and the origin (measured in the counterclockwise direction), then we call \(r\) and \(\theta\) the polar coordinates of \((x, y)\). Also recall that

\[
r = \sqrt{x^2 + y^2},
\]

\[
\tan(\theta) = \frac{y}{x} \text{ (provided } x \neq 0),
\]

\[
x = r \cos(\theta),
\]

and

\[
y = r \sin(\theta).
\]

It follows that if \(z = x + iy\) is a complex number and \(r\) and \(\theta\) are the polar coordinates of \((x, y)\), then

\[
z = r(\cos(\theta) + i \sin(\theta)).
\]

**Example 4.1.** If \(z = 1 + i\), then \(r = \sqrt{2}\) and we may take \(\theta = \frac{\pi}{4}\). That is,

\[
z = \sqrt{2} \left( \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right).
\]
Note, however, that \( \theta \) is not unique. In particular, we could have used \( \theta = \frac{9\pi}{4} \) or \( \theta = -\frac{7\pi}{4} \). In general, any of the values
\[
\frac{\pi}{4} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots,
\]
would work.

We call a given value of the polar coordinate \( \theta \) an \textit{argument} of \( z \) and denote the set of all possible arguments of \( z \) by \( \arg z \). We call the value \( \theta \) of \( \arg z \) for which \( -\pi < \theta \leq \pi \) the \textit{principal value} of \( \arg z \) and denote it by \( \text{Arg} z \).

\textbf{Example 4.2.} We have
\[
\arg(-2 - 2i) = -\frac{3\pi}{4} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots,
\]
and
\[
\text{Arg}(-2 - 2i) = -\frac{3\pi}{4}.
\]

\textbf{Example 4.3.} \( \text{Arg}(-3) = \pi \).

\section*{4.2 Euler’s formula}

We will explain this more carefully later on, but for now we introduce the notation
\[
e^{i\theta} = \exp(i\theta) = \cos(\theta) + i\sin(\theta)
\]
in order to provide a compact way to denote complex numbers in polar form. That is, if \( z \) is a complex number with polar coordinates \( r \) and \( \theta \), we may write
\[
z = re^{i\theta}.
\]
We will see that this agrees with the exponentional function from real variable calculus, but for now we must remember that it is only notation.

\textbf{Example 4.4.} We may now write
\[
1 + i = \sqrt{2}e^{\frac{\pi}{4}}
\]
and
\[
-2 - 2i = 2\sqrt{2}e^{-i\frac{3\pi}{4}}.
\]
Example 4.5. Note that $e^{i\pi} = -1$.

Example 4.6. For $\theta$ going from 0 to $2\pi$,

$$z = 4e^{i\theta}$$

is a parametrization of the circle of radius 4 centered at the origin.

More generally, for a fixed complex number $z_0$ and real number $R$,

$$z = z_0 + Re^{i\theta},$$

$0 \leq \theta \leq 2\pi$, parametrizes a circle of radius $R$ with center at $z_0$.

Proposition 4.1. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are two complex numbers, then

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$$

and, if $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

Proof. The first result follows from noting that

$$e^{i\theta_1}e^{i\theta_2} = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$$

$$= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))$$

$$+ i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))$$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

For the second, first note that

$$\frac{e^{i\theta}}{e^{-i\theta}} = \frac{1}{e^{i\theta}} = \frac{1}{e^{i\theta}}e^{-i\theta} = \frac{e^{-i\theta}}{e^0} = e^{-i\theta}.$$

It now follows that

$$\frac{z_1}{z_2} = \frac{z_1z_2^{-1}}{z_2^{-1}} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$
As a consequence of this proposition,

$$\arg(z_1 z_2) = \arg z_1 + \arg(z_2),$$

$$\arg(z^{-1}) = -\arg(z),$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Moreover, note that we now have a geometric interpretation of complex multiplication: Multiplying $z$ by $w$ rotates $z$ by an angle $\text{Arg}\ w$ and stretches the result by a factor of $|w|$.

**Example 4.7.** If $z = \frac{1 - i}{2 + 2i}$, then

$$\arg z = \arg(1 - i) - \arg(2 + 2i) = -\frac{\pi}{4} - \frac{\pi}{4} + 2n\pi = -\frac{\pi}{2} + 2n\pi,$$

$n = 0, \pm 1, \pm 2, \ldots$. Hence

$$\text{Arg} z = -\frac{\pi}{2}.$$

Indeed,

$$z = \frac{1 - i}{2 + 2i} \cdot \frac{2 - 2i}{2 - 2i} = \frac{4i}{8} = -\frac{1}{2}i.$$

### 4.3 DeMoivre’s formula

If $z = re^{i\theta}$ and $n$ is a postive integer, then

$$z^n = re^{i\theta} \cdot re^{i\theta} \cdots re^{i\theta} = r^n e^{in\theta}.$$

Using our results about reciprocals, the result also holds when $n$ is a negative integer. Moreover, if we use the convention that $z^0 = 1$, then

$$z^0 = r^0 (\cos(0) + i \sin(0)) = r^0 e^{i\cdot0}.$$

Hence we have

$$z^n = r^n e^{i\cdot \theta} \text{ for } n = 0, \pm 1, \pm 2, \ldots.$$

In the particular case when $r = 1$ this gives us

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta) \text{ for } n = 0, \pm 1, \pm 2, \ldots,$$

which we call *de Moivre’s formula*. 
Example 4.8. If \( z = 1 + \sqrt{3}i \), then \(|z| = 2\) and \(\arg z = \frac{\pi}{3}\). Hence

\[ z^9 = (1 + \sqrt{3}i)^9 = (2e^{\frac{\pi}{3}})^9 = 2^9 e^{3\pi} = -512. \]

Figure 4.1 displays \( z, z^2, \ldots, z^9 \).