Lecture 37:
Laurent Series

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37.1 Laurent’s theorem

The following result is known as Laurent’s theorem.

Theorem 37.1. Suppose \( z_0 \in \mathbb{C} \), \( f \) is analytic in the domain
\[
D = \{ z \in \mathbb{C} : R_1 < |z - z_0| < R_2 \},
\]
and \( C \) is any positively oriented, simple closed contour in \( D \), with \( z_0 \) in the
interior of \( C \). Then, for any \( z \in D \),
\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},
\]
where
\[
a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,
\]
\( n = 0, 1, 2, \ldots \), and
\[
b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,
\]
\( n = 1, 2, 3, \ldots \).

Note that we could write the series above as
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n,
\]
where
\[ c_n = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z-z_0)^{n+1}}, \]

\[ n = 0, \pm 1, \pm 2, \ldots \] Moreover, note if \( f \) is analytic on the entire disk \(|z-z_0|<R_2\), then \( b_n = 0 \) for all \( n \) and

\[ a_n = \frac{f^{(n)}(z_0)}{n!} \]

for \( n = 0, 1, 2, \ldots \). That is, in this case Laurent’s theorem reduces to Taylor’s theorem.

**Proof.** We will assume \( z_0 = 0 \). The general case follows from a translation, as it did in the proof of Taylor’s theorem. Choose \( R_1 < r_1 < R_2 \) and \( r_1 < r_2 < R_2 \) so that the annular region \( r_1 < |w| < r_2 \) contains both \( z \) and \( C \). Let \( C_1 \) be the circle \(|z| = r_1\) and let \( C_2 \) be the circle \(|z| = r_2\). Let \( \gamma \) be a circle centered at \( z \) with radius smaller than both \( r_2 - |z| \) and \(|z| - r_1\). Give \( C_1, C_2 \), and \( \gamma \) positive orientations.

We now have, from extensions to the Cauchy-Goursat theorem, that

\[ \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds - \int_{\gamma} \frac{f(s)}{s-z} ds = 0. \]

By the Cauchy integral formula,

\[ \int_{\gamma} \frac{f(s)}{s-z} ds = 2\pi if(z), \]

and so

\[ \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds = 2\pi if(z). \]

That is,

\[ f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds - \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds \]
\[ = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z-s} ds. \]

We now proceed as we did in the proof of Taylor’s theorem. We first note that, for any positive integer \( N \),

\[ \frac{1}{s-z} = \frac{1}{s} \frac{1}{1 - \frac{z}{s}} \]

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\[ \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds = \sum_{n=0}^{N-1} \left( \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \right) z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s - z)s^N} ds + \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s - z)s^N}, \]

from which it follows that

\[ \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds = \sum_{n=0}^{N-1} \left( \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \right) z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s - z)s^N} ds = \sum_{n=0}^{N-1} a_n z^n + \rho_N(z), \]

where

\[ a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \]

and

\[ \rho_N(z) = \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s - z)s^N} ds. \]

Similarly, interchanging \( z \) and \( s \), we see that, for any positive integer \( N \),

\[ \frac{1}{z - s} = \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{s^N}{(z - s)z^N} = \sum_{n=1}^{N} \frac{s^{n-1}}{z^n} + \frac{s^N}{(z - s)z^N}, \]

from which it follows that

\[ \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z - s} ds = \sum_{n=1}^{N} \left( \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{n-1}} ds \right) \frac{1}{z^n} + \frac{1}{2\pi iz^N} \int_{C_1} \frac{s^{N}f(s)}{z - s} ds = \sum_{n=1}^{N} b_n \frac{z^n}{z^n} + \sigma_N(z), \]

where

\[ b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{n-1}} ds \]

and

\[ \sigma_N(z) = \frac{1}{2\pi iz^N} \int_{C_1} \frac{s^{N}f(s)}{z - s} ds. \]
Now let $M_1$ be the maximum value of $|f(s)|$ on $C_1$, let $M_2$ be the maximum value of $|f(s)|$ on $C_2$, and let $r = |z|$. Then, for $s$ on $C_2$,

$$|s - z| \geq ||s| - |z|| = r_2 - r,$$

and so

$$\left| \frac{f(s)}{(s - z)s^N} \right| \leq \frac{M_2}{(r_2 - r)r_2^N}.$$

Hence

$$|\rho_N(z)| = \left| \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s - z)s^N} ds \right| \leq \frac{r^N}{2\pi} \cdot \frac{M_2}{(r_2 - r)r_2^N} \cdot 2\pi r_2 = \frac{M_2 r_2}{r_2 - r} \cdot \left( \frac{r}{r_2} \right)^N.$$

Since $\frac{r}{r_2} < 1$,

$$\lim_{N \to \infty} \rho_N(z) = 0.$$

For $s$ on $C_2$,

$$|z - s| \geq ||z| - |s|| = r - r_1,$$

and so

$$\left| \frac{s^N f(s)}{z - s} \right| \leq \frac{r_1^N M_1}{r - r_1}.$$

Hence

$$|\sigma_N(z)| = \left| \frac{1}{2\pi i z^N} \int_{C_1} \frac{s^N f(s)}{(s - z)} ds \right| \leq \frac{1}{2\pi r^N} \cdot \frac{r_1^N M_1}{r - r_1} \cdot 2\pi r_1 = \frac{M_1 r_1}{r - r_1} \cdot \left( \frac{r_1}{r} \right)^N.$$

Since $\frac{r_1}{r} < 1$,

$$\lim_{N \to \infty} \sigma_N(z) = 0.$$

Thus we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$
Finally, we note that, since $f$ is analytic in $D$,

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s^{n+1}} ds$$

for $n = 0, 1, 2, \ldots$, and

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s^{-n+1}} ds$$

for $n = 1, 2, 3, \ldots$. This completes the proof when $z_0 = 0$. When $z_0 \neq 0$, define $g(z) = f(z + z_0)$ and proceed as in the proof of Taylor’s theorem. \qed