34.1 Sequences

Definition 34.1. We say an infinite sequence $z_1, z_2, \ldots, z_n$ of complex numbers has a limit $z$ if for every $\epsilon > 0$ there exists a positive integer $n_0$ such that

$$|z_n - z| < \epsilon$$

whenever $n > n_0$, in which case we write

$$\lim_{n \to \infty} z_n = z$$

and we say the sequence converges. If a sequence does not converge, we say it diverges.

As with limits of functions, a sequence can have at most one limit. Moreover, if $z_n = x_n + iy_n$ and $z = x + iy$, where $x_n, y_n, x, y \in \mathbb{R}$, then

$$\lim_{n \to \infty} z_n = z$$

if and only if both

$$\lim_{n \to \infty} x_n = x$$

and

$$\lim_{n \to \infty} y_n = y.$$ 

The proofs of these results parallel the corresponding proof for limits of functions.
Example 34.1. Suppose
\[ z_n = \frac{3}{n^2} + i \left(1 + \frac{1}{n^2}\right). \]

Then
\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} \frac{3}{n^2} + i \lim_{n \to \infty} \left(1 + \frac{1}{n^2}\right) = i. \]

We could verify this limit from the definition as well by first noting that
\[ |z_n - i| = \left| \frac{3}{n^2} + i \frac{1}{n^2} \right| = \frac{\sqrt{10}}{n^2}, \]
and so, given \( \epsilon > 0 \), \( |z_n - i| < \epsilon \) whenever
\[ n > \frac{\sqrt{10}}{\sqrt{\epsilon}}. \]

34.2 Series

Definition 34.2. Given an infinite sequence \( z_1, z_2, z_3, \ldots \), let
\[ S_N = z_1 + z_2 + \cdots + z_N. \]

We call the sequence \( S_1, S_2, S_3, \ldots \) an infinite series, which we denote
\[ \sum_{n=1}^{\infty} z_n. \]

We call \( S_N \) a partial sum. If \( S_N \) converges with \( S = \lim_{N \to \infty} S_N \), then we say \( \sum_{n=1}^{\infty} z_n \) converges and write
\[ \sum_{n=1}^{\infty} z_n = S. \]

If \( S_N \) does not converge, we say \( \sum_{n=1}^{\infty} z_n \) diverges.

Suppose \( z_n = x_n + iy_n \) and \( S = X + iY \). Then it follows from previous results that
\[ \sum_{n=1}^{\infty} z_n = S \]
if and only if 
\[
\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{i=1}^{\infty} y_i = Y.
\]

**Proposition 34.1.** If \( \sum_{n=1}^{\infty} z_n \) converges, then \( \lim_{n \to \infty} z_n = 0 \).

**Proof.** Let \( S = \sum_{n=1}^{\infty} z_n \) and \( S_N = \sum_{n=1}^{N} z_n \). Then
\[
\lim_{N \to \infty} z_N = \lim_{N \to \infty} (S_N - S_{N-1}) = S - S = 0.
\]

\[\square\]

**Definition 34.3.** We say an infinite series
\[
\sum_{n=1}^{\infty} z_n
\]

is **absolutely convergent** if the infinite series
\[
\sum_{n=1}^{\infty} |z_n|
\]

converges.

**Proposition 34.2.** If \( \sum_{n=1}^{\infty} z_n \) is absolutely convergent, then it is convergent.

**Proof.** Suppose \( z_n = x_n + iy_n \) and
\[
\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}
\]

converges. Since
\[
|x_n| \leq \sqrt{x_n^2 + y_n^2}
\]

and
\[
|y_n| \leq \sqrt{x_n^2 + y_n^2},
\]

it follows, by the comparison test, that both
\[
\sum_{n=1}^{\infty} |x_n|
\]
and
\[ \sum_{n=1}^{\infty} |y_n| \]
converge. Hence, by a result from calculus, both \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \) converge. Thus \( \sum_{n=1}^{\infty} z_n \) converges.

**Definition 34.4.** Given a complex numbers \( a_0, a_1, a_2, \ldots \), and \( z_0 \), we call an infinite series of the form
\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots
\]
a **power series**

**Example 34.2.** Let \( z \in \mathbb{C} \) and consider the power series
\[
\sum_{n=0}^{\infty} z^n.
\]
If
\[
S_N(z) = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \cdots + z^{N-1},
\]
then, from an earlier homework problem,
\[
S_N(z) = \frac{1 - z^N}{1 - z},
\]
when \( z \neq 1 \). Let
\[
S(z) = \frac{1}{1 - z}.
\]
If we let
\[
\rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1 - z},
\]
then
\[
|\rho_N(z)| = \frac{|z|^N}{|1 - z|}.
\]
It follows that
\[
\lim_{N \to \infty} |\rho_N(z)| = 0
\]
if and only if $|z| < 1$. That is,

$$
\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}
$$

if and only if $|z| < 1$. Put another way, the power series

$$1 + z + z^2 + \cdots$$

converges to $\frac{1}{1-z}$ for all $z$ in the open disk $|z| < 1$ and for no other points in the plane.