33.1 Maximum of the modulus

Lemma 33.1. Suppose \( f \) is analytic in the \( \epsilon \) neighborhood \( U \) of \( z_0 \). If \( |f(z)| \leq |f(z_0)| \) for all \( z \in U \), then \( f(z) \) is constant on \( U \).

Proof. Let \( 0 < \rho < \epsilon \) and let \( C_\rho \) be the circle \( |z - z_0| = \rho \). By the Cauchy integral formula, we know that

\[
f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz.
\]

If we parametrize \( C_\rho \) by \( z = z_0 + \rho e^{it}, 0 \leq t \leq 2\pi \), then

\[
f(z_0) = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it} dt = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + \rho e^{it}) dt.
\]

(Note that this means that \( f(z_0) \) is the average of the values of \( f(z) \) on \( C_\rho \).) Hence

\[|f(z_0)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(z_0 + \rho e^{it})| dt.
\]

However, the assumption \( |f(z_0)| \geq |f(z)| \) for all \( z \in U \) implies that

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |f(z_0 + \rho e^{it})| dt \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(z_0)| dt = |f(z_0)|.
\]
Hence we must in fact have

\[ |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| dt. \]

It follows that

\[ 0 = |f(z_0)| - \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt - \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} (|f(z_0)| - |f(z_0 + \rho e^{it})|) dt. \]

Since \( |f(z_0)| - |f(z_0 + \rho e^{it})| \) is a continuous function of \( t \) and

\[ |f(z_0)| - |f(z_0 + \rho e^{it})| \geq 0 \]
for all \( t \in [0, 2\pi] \), it follows that

\[ |f(z_0)| = |f(z_0 + \rho e^{it})| \]
for all \( t \in [0, 2\pi] \); that is, \( |f(z_0)| = |f(z)| \) for all \( z \in C_\rho \). Since \( \rho \) was arbitrary, it follows that \( |f(z_0)| = |f(z)| \) for all \( z \in U \). Finally, using a previous homework problem, we may now conclude that \( f(z) = f(z_0) \) for all \( z \in U \).

With the lemma, we may now prove the maximum modulus principle.

**Theorem 33.1.** Suppose \( D \subset \mathbb{C} \) is a domain and \( f : D \to \mathbb{C} \) is analytic in \( D \). If \( f \) is not a constant function, then \( |f(z)| \) does not attain a maximum on \( D \).

**Proof.** Suppose, to the contrary, that there exists a point \( z_0 \in D \) for which \( |f(z_0)| \geq |f(z)| \) for all other points \( z \in D \). We will show that \( f \) must then be a constant function. Let \( w \) be any other point in \( D \) and consider a polygonal path \( L \) from \( z_0 \) to \( w \). If \( D \) is not the entire plane, let \( \delta \) be the minimum distance from \( L \) to the boundary of \( D \); if \( D \) is the entire plane, let \( \delta = 1 \). Consider a finite sequence of points \( z_0, z_1, z_2, \ldots, z_n = w \) with \( z_k \in L \) and \( |z_k - z_{k-1}| < \delta \) for \( k = 1, 2, \ldots, n \). For example, we might construct these points by moving a distance \( \frac{\delta}{2} \) along \( L \) from one to the next. Let \( U_k \) be the
δ neighborhood of $z_k$, $k = 0, 1, 2, \ldots, n$. By the lemma, $f(z) = f(z_0)$ for all $z \in U_0$. Since $z_1 \in U_0$, $f(z_1) = f(z_0)$. Then $|f(z_1)|$ is the maximum value of $|f(z)|$ on $U_1$, and so $f(z) = f(z_0)$ for all $z \in U_1$. Since $z_2 \in U_1$, we then have $f(z_2) = f(z_0)$, from which it follows that $f(z) = f(z_0)$ for all $z \in U_2$. Continuing in this manner, we eventually reach $f(w) = f(z_n) = f(z_0)$. Since $w$ was an arbitrary point in $D$, it follows that $f(z) = f(z_0)$ for all $z \in D$.

**Corollary 33.1.** Suppose $R \subset \mathbb{C}$ is a closed bounded region. If $f : R \rightarrow \mathbb{C}$ is continuous on $R$, analytic on the interior of $R$, and not constant, then the maximum value of $|f(z)|$ is attained at a point (or points) on the boundary of $R$ and never at points in the interior of $R$. Moreover, if we write

$$f(x + iy) = u(x, y) + iv(x, y),$$

then the maximum value of $u(x, y)$ is attained at a point (or points) on the boundary of $R$ and never at points in the interior of $R$.

**Proof.** The first part follows from the fact that a continuous function on a closed bounded set attains a maximum value, and from the maximum modulus principle this value cannot be attained in the interior of $R$. The second part follows from the observation that the modulus of the function

$$g(z) = e^{f(z)}$$

is

$$|g(z)| = e^{u(x, y)}.$$