25.1 Contour integrals

Definition 25.1. Suppose $z(t), a \leq t \leq b$, parametrizes a contour $C$ and $f$ is complex-valued function for which $f(z(t))$ is piecewise continuous on $[a, b]$. We call

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

the contour integral of $f$ along $C$.

Example 25.1. We will evaluate

$$\int_C z^2dz$$

where $C$ is parametrized by $z(t) = e^{it}, 0 \leq t \leq \pi$. We have

$$\int_C z^2dz = \int_0^\pi e^{i2t}(ie^{it})dt$$

$$= i \int_0^\pi e^{3it}dt$$

$$= \frac{1}{3} e^{3it} \bigg|_0^\pi$$

$$= \frac{1}{3}(-1 - 1)$$

$$= -\frac{2}{3}$$
From our earlier discussion of integrals, it follows easily that if \( c \in \mathbb{C} \) is a constant and \( f \) and \( g \) are complex-valued functions, then

\[
\int_C cf(z)\,dz = c \int_C f(z)\,dz
\]

and

\[
\int_C (f(z) + g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.
\]

Also, if \( C_1 \) and \( C_2 \) are two contours with the terminal point of \( C_1 \) the same as the initial point of \( C_2 \), and we let \( C \) denote the contour formed by \( C_1 \) and \( C_2 \) together, then

\[
\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.
\]

We may denote \( C \) by \( C_1 + C_2 \), in which case we write

\[
\int_{C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.
\]

Note that if \( z(t), a \leq t \leq b \), parametrizes \( C \), then

\[
w(t) = z(-t), \quad -b \leq t \leq -a,
\]

parametrizes \( C \) with the opposite orientation. We denote this contour by \(-C\). It follows that, using the substitution \( s = -t \),

\[
\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(w(t))w'(t)\,dt
\]

\[
= - \int_{-b}^{-a} f(z(-t))z'(-t)\,dt
\]

\[
= \int_{b}^{a} f(z(s))z'(s)\,ds
\]

\[
= - \int_{a}^{b} f(z(s))z'(s)\,ds
\]

\[
= - \int_{C} f(z)\,dz.
\]
Note that if $C_1$ and $C_2$ have the same terminal point, then the terminal point of $C_1$ is the same as the initial point of $-C_2$. Hence we may consider the contour $C_1 + (-C_1)$, which we, of course, denote $C_1 - C_2$. We have

$$\int_{C_1-C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz.$$ 

### 25.2 Examples

**Example 25.2.** Let $f(x + iy) = xy + i(x + y)$ and let $C$ be the triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$, oriented in the counterclockwise direction. To evaluate $\int_C f(z)dz$, we will write $C$ as $C_1 + C_2 - C_3$, where $C_1$ has parametrization

$$z = x, 0 \leq x \leq 1,$$

$C_2$ has parametrization

$$z = 1 + iy, 0 \leq y \leq 1,$$

and $C_3$ has parametrization

$$z = x + ix, 0 \leq x \leq 1.$$

Then

$$\int_{C_1} f(z)dz = \int_0^1 ix \, dx = \frac{i}{2},$$

$$\int_{C_2} f(z)dz = \int_0^1 (y + i(1 + y)) \, idy = -\frac{3}{2} + i\frac{1}{2},$$

and

$$\int_{C_3} f(z)dz = \int_0^1 (x^2 + i2x)(1 + i) \, dx = \left(\frac{1}{3} + i\right) (1 + i) = -\frac{2}{3} + i\frac{4}{3}.$$ 

Hence

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz - \int_{C_3} f(z)dz$$

$$= \frac{i}{2} - \frac{3}{2} + i\frac{1}{2} + \frac{2}{3} - i\frac{4}{3}$$

$$= -\frac{5}{6} - \frac{1}{3}i.$$
Note that
\[
\int_{C_1+C_2} f(z)dz = \frac{i}{2} - \frac{3}{2} + i \frac{1}{2} = -\frac{3}{2} + i \neq \int_{C_3} f(z)dz,
\]
even though \(C_1 + C_2\) and \(C_3\) have the same initial and final points. Hence, although the value of a contour integral does not depend on the specific parametrization of a given arc (see the homework), it may depend on the curve chosen to get from the initial point to the final point.

**Example 25.3.** Let \(C\), with parametrization \(z(t), a \leq t \leq b\), be a smooth arc and let \(z_1 = z(a)\) and \(z_2 = z(b)\). Then
\[
\int_C z^2 dz = \int_a^b (z(t))^2 z'(t) dt
\]
\[
= \frac{z(t)^3}{3} \bigg|_a^b
\]
\[
= \frac{z_2^3 - z_1^3}{3}.
\]
Note that this means that this contour integral is independent of the particular curve starting at \(z_1\) and ending at \(z_2\). For example, for any curve \(C\) starting at \(z_1 = 1\) and ending at \(z_2 = -1\), we have
\[
\int_C z^2 dz = \frac{(-1)^3 - 1^3}{3} = -\frac{2}{3}.
\]
Recall that this is the result we obtained in our first example for the particular arc \(z = e^{it}, 0 \leq t \leq \pi\). This result will also hold for any contour \(C\). Moreover, it follows that if \(C\) is a closed contour, then
\[
\int_C z^2 dz = 0.
\]

**Example 25.4.** Let \(C\) be the unit circle with parametrization \(z = e^{it}\). Then
\[
\int_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it}(ie^{it}) dt = \int_0^{2\pi} i dt = 2\pi i.
\]
Does this contradict our observations in the previous example and the fact that
\[
\frac{d}{dz} \log(z) = \frac{1}{z}?