22.1 Inverse trigonometric functions

Given $z \in \mathbb{C}$, we would like to find $w \in \mathbb{C}$ such that $z = \sin(w)$. That is, we want to solve

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$  

It follows that

$$2iz e^{iw} = e^{2iw} - e^0 = (e^{iw})^2 - 1,$$

or, equivalently,

$$(e^{iw})^2 - 2iz e^{iw} - 1 = 0.$$  

Using the quadratic formula, we find that

$$e^{iw} = \frac{2iz + \left(-4z^2 + 4\right)^{\frac{1}{2}}}{2} = iz + (1 - z^2)^{\frac{1}{2}}.$$  

Hence

$$w = -i \log \left(iz + (1 - z^2)^{\frac{1}{2}}\right).$$

It follows that we may define the inverse sine function, or arcsine function, as

$$\sin^{-1}(z) = -i \log \left(iz + (1 - z^2)^{\frac{1}{2}}\right),$$

which we may also denote as $\arcsin(z)$. Note that this is a multi-valued function, with values depending both on the branch of the square root function and the branch of the logarithmic function chosen.
Example 22.1. Note that
\[ \sin^{-1}(1) = -i \log(i) = -i \left( i \left( \frac{\pi}{2} + 2n\pi \right) \right) = \frac{\pi}{2} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots, \]
as we should expect.

Example 22.2. When \( z = 0 \), then \( (1 - z^2)^{1/2} = \pm 1 \). Since
\[ \log(1) = 2n\pi i, n = 0, \pm 1, \pm 2, \ldots, \]
and
\[ \log(-1) = i(\pi + 2n\pi) = (2n + 1)i, n = 0, \pm 1, \pm 2, \ldots, \]
we have
\[ \sin^{-1}(0) = n\pi, n = 0, \pm 1, \pm 2, \ldots, \]
again, as we should expect.

Example 22.3. If \( z = 2 \), then \( (1 - z^2)^{1/2} = \pm \sqrt{3}i \),
\[ \log(2i + \sqrt{3}i) = \log(i(2 + \sqrt{3})) = \ln(2 + \sqrt{3}) + i \left( \frac{\pi}{2} + 2n\pi \right), n = 0, \pm 1, \pm 2, \ldots, \]
and
\[ \log(2i - \sqrt{3}i) = \log(i(2 - \sqrt{3})) = \ln(2 - \sqrt{3}) + i \left( \frac{\pi}{2} + 2n\pi \right), n = 0, \pm 1, \pm 2, \ldots, \]
Now \( (2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 3 = 1 \), so
\[ 2 - \sqrt{3} = \frac{1}{2 + \sqrt{3}}. \]
Hence
\[ \log(2i - \sqrt{3}i) = -\ln(2 + \sqrt{3}) + i \left( \frac{\pi}{2} + 2n\pi \right), n = 0, \pm 1, \pm 2, \ldots. \]
Hence
\[ \sin^{-1}(2) = \frac{\pi}{2} + 2n\pi \pm i \ln(2 + \sqrt{3}), n = 0, \pm 1, \pm 2, \ldots. \]
When specific branches of the square root and the logarithmic functions are chosen, we may differentiate $\sin^{-1}(z)$:

$$
\frac{d}{dz} \sin^{-1}(z) = \frac{d}{dz} \left( -i \log \left( iz + (1 - z^2)^{\frac{1}{2}} \right) \right)
$$

$$
= \frac{-i}{iz + (1 - z^2)^{\frac{1}{2}}} \left( i + \frac{-2z}{2(1 - z^2)^{\frac{1}{2}}} \right)
$$

$$
= \frac{-i}{iz + (1 - z^2)^{\frac{1}{2}}} \left( \frac{i(1 - z^2)^{\frac{1}{2}} - z}{(1 - z^2)^{\frac{1}{2}}} \right)
$$

$$
= \frac{1}{(1 - z^2)^{\frac{1}{2}}}
$$

Note that the result depends on the branch of the square root function chosen, but not on the particular branch of the logarithmic function. Also, note that $\sin^{-1}(z)$ has singular points at $\pm 1$.

We may define, in an analogous manner, inverse functions for the remaining circular trigonometric functions. The most important of these are

$$
\cos^{-1}(z) = -i \log \left( z + i(1 - z^2)^{\frac{1}{2}} \right)
$$

and

$$
\tan^{-1}(z) = \frac{i}{2} \log \frac{i + z}{i - z},
$$

with derivatives

$$
\frac{d}{dz} \cos^{-1}(z) = -\frac{1}{(1 - z^2)^{\frac{1}{2}}}
$$

and

$$
\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1 + z^2}.
$$

22.2 Inverse hyperbolic functions

Given $z \in \mathbb{C}$, we would like to find $w \in \mathbb{C}$ such that $z = \tanh(w)$. That is, we want to solve

$$
\frac{e^w - e^{-w}}{e^w + e^{-w}} = z
$$

It follows that

$$
z e^{2w} + z = e^{2w} - 1,
$$

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or, equivalently,

\[ e^{2w} = \frac{1 + z}{1 - z}. \]

Hence

\[ w = \frac{1}{2} \log \frac{1 + z}{1 - z}. \]

Thus we define the inverse hyperbolic tangent function by

\[ \tanh^{-1}(z) = \frac{1}{2} \log \frac{1 + z}{1 - z}. \]

We find the other inverse hyperbolic trigonometric functions in a similar manner. The most important of these are

\[ \sinh^{-1}(z) = \log \left( z + (z^2 + 1)^{\frac{1}{2}} \right) \]

and

\[ \cosh^{-1}(z) = \log \left( z + (z^2 - 1)^{\frac{1}{2}} \right). \]

The derivatives are

\[ \frac{d}{dz} \sinh^{-1}(z) = \frac{1}{(z^2 + 1)^{\frac{1}{2}}}, \]

\[ \frac{d}{dz} \cosh^{-1}(z) = \frac{1}{(z^2 - 1)^{\frac{1}{2}}}, \]

and

\[ \frac{d}{dz} \tanh^{-1}(z) = \frac{1}{1 - z^2}. \]