20.1 Defining sine and cosine

Recall that if \( x \in \mathbb{R} \), then
\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.
\]

Note what happens if we (somewhat blindly) let \( x = i\theta \):
\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots
= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)
= \cos(\theta) + i\sin(\theta).
\]

This is the motivation for our earlier definition of \( e^{i\theta} \). It now follows that for any \( x \in \mathbb{R} \), we have
\[
e^{ix} = \cos(x) + i\sin(x) \quad \text{and} \quad e^{-ix} = \cos(x) - i\sin(x),
\]
from which we obtain (by addition)
\[
2\cos(x) = e^{ix} + e^{-ix}
\]
and (by subtraction)
\[
2i\sin(x) = e^{ix} - e^{-ix}.
\]
Hence we have
\[ \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \]
and
\[ \sin(x) = \frac{e^{ix} - e^{-ix}}{2}, \]
which motivate the following definitions.

**Definition 20.1.** For any complex number \( z \), we define the **sine** function by
\[ \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \]
and the **cosine** function by
\[ \cos(z) = \frac{e^{iz} + e^{-iz}}{2}. \]

The following proposition is immediate from the properties of analytic functions and the fact that \( e^z \) is an entire function.

**Proposition 20.1.** Both \( \sin(z) \) and \( \cos(z) \) are entire functions.

**Proposition 20.2.** For all \( z \in \mathbb{C} \),
\[ \frac{d}{dz} \sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz} \cos(z) = -\sin(z). \]

**Proof.** We have
\[ \frac{d}{dz} \sin(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos(z) \]
and
\[ \frac{d}{dz} \cos(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z). \]

\[ \square \]

### 20.2 Properties of sine and cosine

**Proposition 20.3.** For any \( z \in \mathbb{C} \),
\[ \sin(-z) = -\sin(z) \quad \text{and} \quad \cos(-z) = \cos(z). \]
Proof. We have
\[
\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i} = -\sin(z)
\]
and
\[
\cos(-z) = \frac{e^{-iz} + e^{iz}}{2} = \cos(z).
\]

Proposition 20.4. For any \(z_1, z_2 \in \mathbb{C}\),
\[2 \sin(z_1) \cos(z_2) = \sin(z_1 + z_2) + \sin(z_1 - z_2).\]
Proof. We have
\[
2 \sin(z_1) \cos(z_2) = 2 \left( \frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left( \frac{e^{iz_2} + e^{-iz_2}}{2} \right)
= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1+z_2)} - e^{-i(z_1-z_2)}}{2i}
= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} + \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i}
= \sin(z_1 + z_2) + \sin(z_1 - z_2).
\]

Proposition 20.5. For any \(z_1, z_1 \in \mathbb{C}\),
\[
\sin(z_1 + z_2) = \sin(z_1) \cos(z_2) + \cos(z_1) \sin(z_2)
\]
and
\[
\cos(z_1 + z_2) = \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2).
\]
Proof. From the previous result, we have
\[
2 \sin(z_1) \cos(z_2) = \sin(z_1 + z_2) + \sin(z_1 - z_2)
\]
and
\[
2 \sin(z_2) \cos(z_1) = \sin(z_1 + z_2) - \sin(z_1 - z_2),
\]
from which we obtain the first identify by addition. It now follows that if
\[
f(z) = \sin(z + z_2),
\]
then
\[ f(z) = \sin(z) \cos(z_2) + \cos(z) \sin(z_2) \]
as well. Hence
\[ \cos(z_1 + z_2) = f'(z_1) = \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2). \]

The following identities follow immediately from the previous propositions.

**Proposition 20.6.** For any \( z \in \mathbb{C} \),
\[
\begin{align*}
\sin^2(z) + \cos^2(z) & = 1, \\
\sin(2z) & = 2 \sin(z) \cos(z), \\
\cos(2z) & = 2 \cos^2(z) - \sin^2(z), \\
\sin \left( z + \frac{\pi}{2} \right) & = \cos(z), \\
\sin \left( z - \frac{\pi}{2} \right) & = -\cos(z), \\
\cos \left( z + \frac{\pi}{2} \right) & = -\sin(z), \\
\cos \left( z - \frac{\pi}{2} \right) & = \sin(z), \\
\sin(z + \pi) & = -\sin(z), \\
\cos(z + \pi) & = -\cos(z), \\
\sin(z + 2\pi) & = \sin(z),
\end{align*}
\]
and
\[ \cos(z + 2\pi) = \cos(z). \]

**Proposition 20.7.** For any \( z = x + iy \in \mathbb{C} \),
\[ \sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y) \]
and
\[ \cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y). \]
Proof. We first note that
\[
\cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh(y)
\]
and
\[
\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i\frac{e^{y} - e^{-y}}{2} = i \sinh(y).
\]
Hence
\[
\sin(x + iy) = \sin(x) \cos(iy) + \sin(iy) \cos(x) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]
and
\[
\cos(x + iy) = \cos(x) \cos(iy) - \sin(x) \sin(iy) = \cos(x) \cosh(y) - i \sin(x) \sinh(y).
\]

It now follows (see the homework) that
\[
|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)
\]
and
\[
|\cos(z)|^2 = \cos^2(x) + \sinh^2(y).
\]
Since \(\sinh(y) = 0\) if and only if \(y = 0\), we see that \(\sin(z) = 0\) if and only if \(y = 0\) and \(x = n\pi\) for some \(n = 0, \pm 1, \pm 2, \ldots\), and \(\cos(z) = 0\) if and only if \(y = 0\) and \(x = \frac{\pi}{2} + n\pi\) for some \(n = 0, \pm 1, \pm 2, \ldots\). That is, \(\sin(z) = 0\) if and only if
\[
z = n\pi, n = 0, \pm 1, \pm 2, \ldots,
\]
and \(\cos(z) = 0\) if and only if
\[
z = \frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2, \ldots.
\]

20.3 The other trigonometric functions
The rest of the trigonometric functions are defined as usual:
\[
\tan(z) = \frac{\sin(z)}{\cos(z)}.
\]
\[
\cot(z) = \frac{\cos(z)}{\sin(z)},
\]
\[
\sec(z) = \frac{1}{\cos(z)},
\]
and
\[
\csc(z) = \frac{1}{\sin(z)}.
\]
Using our results on derivatives, it is straightforward to show that
\[
\frac{d}{dz} \tan(z) = \sec^2(z),
\]
\[
\frac{d}{dz} \cot(z) = -\csc^2(z),
\]
\[
\frac{d}{dz} \sec(z) = \sec(z) \tan(z),
\]
and
\[
\frac{d}{dz} \csc(z) = -\csc(z) \cot(z).
\]
In particular, these functions are analytic at all points at which they are defined. As with their real counterparts, they are all periodic, \(\tan(z)\) and \(\cot(z)\) having period \(\pi\) and \(\sec(z)\) and \(\csc(z)\) having period \(2\pi\).