18.1 Branches

Note that
\[ \log(z) = \ln|z| + i \text{Arg} \, z \]
is not continuous for any \( z_0 = x_0 + iy_0 \) with \( y_0 = 0 \) and \( x_0 \leq 0 \) since \( \log(z) \to \ln|x_0| + i\pi \) as \( z = x + iy \) approaches \( z_0 \) with \( y > 0 \) and \( \log(z) \to \ln|x_0| - i\pi \) as \( z = x + iy \) approaches \( z_0 \) with \( y < 0 \). However, if we restrict to \( z = re^{i\theta} \) with \(-\pi < \theta < \pi\) and write \( \log(z) = u(r, \theta) + iv(r, \theta) \), then
\[ u(r, \theta) = \ln(r) \quad \text{and} \quad v(r, \theta) = \theta, \]
and so
\[ u_r(r, \theta) = \frac{1}{r} \quad \text{and} \quad u_\theta(r, \theta) = 0 \]
and
\[ v_r(r, \theta) = 0 \quad \text{and} \quad v_\theta(r, \theta) = 1. \]
Hence
\[ ru_r(r, \theta) = v_\theta(r, \theta) \quad \text{and} \quad u_\theta(r, \theta) = -rv_r(r, \theta). \]
That is, \( u \) and \( v \) satisfy the Cauchy-Riemann equations, and so \( \log(z) \) is analytic in
\[ U = \{ z = re^{i\theta} \in \mathbb{C} : r > 0, -\pi < \theta < \pi \}. \]
Moreover, for all \( z \in U \),
\[
\frac{d}{dz} \log z = e^{-i\theta}(u_r(r, \theta) + iv_r(r, \theta)) = e^{-i\theta} \left( \frac{1}{r} + i \cdot 0 \right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.
\]

More generally, if for any real number \( \alpha \) we restrict \( \log(z) \) to 
\[
\log z = \ln r + i\theta,
\]
where \( z = re^{i\theta}, \ r > 0, \) and \( \alpha < \theta < \alpha + 2\pi \), then \( \log z \) is analytic in
\[
U = \{ z = re^{i\theta} \in \mathbb{C} : r > 0, \alpha < \theta < \alpha + 2\pi \}
\]
with
\[
\frac{d}{dz} \log z = \frac{1}{z}.
\]
We call such a restricted version of \( \log z \) a branch of the multi-valued function \( \log z \), with the restricted version of \( \text{Log} z \) discussed above being the principal branch. We call the origin along with the ray consisting of all points \( z = re^{i\theta} \) for which \( \theta = \alpha \) a branch cut; we call the origin a branch point because it is common to all the branch cuts.

### 18.2 Properties of logarithms

**Proposition 18.1.** For any \( z_1, z_2 \in \mathbb{C} \), with \( z_1 \neq 0 \) and \( z_2 \neq 0 \), then
\[
\log(z_1z_2) = \log(z_1) + \log(z_2)
\]
and
\[
\log \left( \frac{z_1}{z_2} \right) = \log(z_1) - \log(z_2).
\]

**Proof.** We have
\[
\log(z_1z_2) = \ln(|z_1z_2|) + i \arg(z_1z_2)
\]
\[
= \ln(|z_1||z_2|) + i(\arg(z_1) + \arg(z_2))
\]
\[
= (\ln(|z_1|) + i \arg(z_1)) + (\ln(|z_2|) + i \arg(z_2))
\]
\[
= \log(z_1) + \log(z_2).
\]

and
\[
\log \left( \frac{z_1}{z_2} \right) = \ln \left| \frac{z_1}{z_2} \right| + i \arg \left( \frac{z_1}{z_2} \right)
\]
\[
\begin{align*}
&= \ln \left( \left| z_1 \right| \right) + i(\arg(z_1) - \arg(z_2)) \\
&= (\ln(|z_1| + i \arg(z_1)) - (\ln(|z_2| + i \arg(z_2)) \\
&= \log(z_1) - \log(z_2).
\end{align*}
\]

Example 18.1. Let \( z_1 = -2i \) and \( z_2 = -i \). Then

\[
\log(z_1) = \ln(2) + i \left( -\frac{\pi}{2} + 2n\pi \right), \ n = 0, \pm 1, \pm 2, \ldots,
\]

\[
\log(z_2) = i \left( -\frac{\pi}{2} + 2n\pi \right), \ n = 0, \pm 1, \pm 2, \ldots,
\]

and

\[
\log(z_1 z_2) = \log(-2) = \ln(2) + i(\pi + 2n\pi), \ n = 0, \pm 1, \pm 2, \ldots
\]

Clearly,

\[
\log(z_1 z_2) = \log(z_1) + \log(z_2).
\]

However,

\[
\Log(z_1) = \ln(2) - i\frac{\pi}{2},
\]

\[
\Log(z_2) = -i\frac{\pi}{2},
\]

\[
\Log(z_1 z_2) = \Log(-2) = \ln(2) + i\pi,
\]

and so

\[
\Log(z_1) + \Log(z_2) = \ln(2) - i\pi \neq \Log(z_1 z_2).
\]

As a prelude to discussing complex exponents, we note two more properties of logarithms. First, if \( z = re^{i\theta}, \ r > 0 \), then, since \( z = e^{\log(z)} \), we have

\[
z^n = e^{n \log(z)}, \ n = 0, \pm 1, \pm 2, \ldots
\]

Next, if \( n \) is a positive integer, \( \Theta = \text{Arg}(z), \ z = re^{i\Theta} \neq 0 \), then, for \( k = 0, \pm 1, \pm 2, \ldots \),

\[
e^{\frac{1}{n} \log(z)} = e^{\left( \frac{1}{n} \ln(r) + i \frac{\Theta + 2k\pi}{n} \right)} = \sqrt[n]{r} e^{i \left( \frac{\Theta}{n} + \frac{2k\pi}{n} \right)} = z^{\frac{1}{n}}.
\]

This works as well when \( n \) is a negative integer by noting that

\[
z^{\frac{1}{n}} = \left( z^{-\frac{1}{n}} \right)^{-1} = \left( e^{-\frac{1}{n} \log(z)} \right)^{-1} = e^{\frac{1}{n} \log(z)}.
\]