12.1 The derivative

Definition 12.1. Suppose $f$ is defined on a neighborhood of a point $z_0 \in \mathbb{C}$. We say $f$ is differentiable at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, in which case we call

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

the derivative of $f$ at $z_0$.

Note that, letting $\Delta z = z - z_0$, we could also write

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Moreover, if we let $w = f(z)$ and $\Delta w = f(z + \Delta z) - f(z)$, then we may write

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}.$$
**Example 12.1.** Suppose $f(z) = z^n$, where $n$ is a positive integer. Then

$$f(z + \Delta z) - f(z) = (z + \Delta z)^n - z^n$$

$$= (z^n + nz^{n-1}\Delta z + \cdots + nz(\Delta z)^{n-1} + (\Delta z)^n) - z^n$$

$$= nz^{n-1}\Delta z + \cdots + nz(\Delta z)^{n-1} + (\Delta z)^n,$$

so

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = nz^{n-1} + \cdots + nz(\Delta z)^{n-2} + (\Delta z)^{n-1}.$$ 

Hence

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = nz^{n-1}.$$

**Example 12.2.** Let

$$f(z) = |z|^2 = zz^\ast.$$ 

Then

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \Delta \bar{z}) - zz^\ast}{\Delta z}$$

$$= \frac{z\Delta \bar{z} + \bar{z}\Delta z + \Delta z\Delta \bar{z}}{\Delta z}$$

$$= \frac{\bar{z} + \Delta \bar{z} + \Delta z}{\Delta z}.$$

It follows that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} \to \bar{z} + z$$

as $\Delta z \to 0$ along the real axis and

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} \to \bar{z} - z$$

as $\Delta z \to 0$ along the imaginary axis. Hence $f$ is not differentiable at any $z \neq 0$. If $z = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 0,$$

and so $f'(0) = 0$. Note that if we write $f(x + iy) = u(x, y) + iv(x, y)$, then

$$u(x, y) = x^2 + y^2.$$
and
\[ v(x, y) = 0. \]
Hence \( u \) and \( v \) have continuous partial derivatives of all order. This shows that the differentiability of \( u \) and \( v \) does not imply that \( f \) is differentiable. Moreover, note that this also shows that a function may be continuous at a point without being differentiable at that point.

**Proposition 12.1.** If \( f \) is differentiable at \( z_0 \), then \( f \) is continuous at \( z_0 \).

**Proof.** We need to show that
\[ \lim_{z \to z_0} f(z) = f(z_0), \]
or, equivalently, that
\[ \lim_{z \to z_0} (f(z) - f(z_0)) = 0. \]
The latter follows from
\[
\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0) \\
= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) \\
= f'(z_0)(0) \\
= 0.
\]

\[ \square \]

### 12.2 Differentiation formulas

**Proposition 12.2.** If \( c \in \mathbb{C} \) and \( f(z) = c \) for all \( c \in \mathbb{C} \), then \( f'(z) = 0 \) for all \( z \in \mathbb{C} \).

**Proof.** We have
\[
f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \lim_{w \to z} \frac{c - c}{w - z} = 0.
\]

\[ \square \]
Proposition 12.3. If $c \in \mathbb{C}$ and $f$ is differentiable at $z$, then
\[
\frac{d}{dz}(cf(z)) = cf'(z).
\]

Proof. We have
\[
\frac{d}{dz}(cf(z)) = \lim_{w \to z} \frac{cf(w) - cf(z)}{w - z} = c \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = cf'(z).
\]

Proposition 12.4. If $f$ and $g$ are both differentiable at $z$, then
\[
\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z),
\]
\[
\frac{d}{dz}f(z)g(z) = f(z)g'(z) + g(z)f'(z),
\]
and, if $g(z) \neq 0$,
\[
\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}.
\]

Proof. For the first statement, we have
\[
\frac{d}{dz}(f(z) + g(z)) = \lim_{w \to z} \frac{(f(w) + g(w)) - (f(z) + g(z))}{w - z}
\]
\[
= \lim_{w \to z} \left( \frac{f(w) - f(z)}{w - z} + \frac{g(w) - g(z)}{w - z} \right)
\]
\[
= f'(z) + g'(z).
\]

For the second,
\[
\frac{d}{dz}f(z)g(z) = \lim_{w \to z} \frac{f(w)g(w) - f(z)g(z)}{w - z}
\]
\[
= \lim_{w \to z} \frac{f(w)g(w) - f(z)g(w) + f(z)g(w) - f(z)g(z)}{w - z}
\]
\[
= \lim_{w \to z} \left( g(w) \frac{f(w) - f(z)}{w - z} + f(z) \frac{g(w) - g(z)}{w - z} \right)
\]
\[
= g(z)f'(z) + f(z)g'(z).
\]
And for the third,
\[
\frac{d}{dz} f(z) = \lim_{w \to z} \frac{f(w) - f(z)}{g(w) - g(z)}
\]
\[
= \lim_{w \to z} \frac{f(w)g(z) - f(z)g(w)}{g(w)g(z)(w - z)}
\]
\[
= \lim_{w \to z} \frac{f(w)g(z) - f(z)g(z) + f(z)g(z) - f(z)g(w)}{g(w)g(z)(w - z)}
\]
\[
= \lim_{w \to z} \frac{g(z)\left(f(w) - f(z) - f(z)\frac{g(w) - g(z)}{w - z}\right)}{g(w)g(z)}
\]
\[
= \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}.
\]

\[\square\]

**Proposition 12.5.** If \( f \) is differentiable at \( z_0 \) and \( g \) is differentiable at \( f(z_0) \), then
\[
(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).
\]

**Proof.** Let \( w_0 = f(z_0) \) and choose \( \epsilon > 0 \) so that \( g \) is defined on the \( \epsilon \) neighborhood of \( w_0 \). Call this neighborhood \( W \). For \( w \in W \), define
\[
\Phi(w) = \begin{cases} 
  \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0), & \text{if } w \neq w_0, \\
  0, & \text{if } w = w_0.
\end{cases}
\]
Note that
\[
\lim_{w \to w_0} \Phi(w) = g'(w_0) - g'(w_0) = 0 = \Phi(w_0),
\]
so \( \Phi \) is continuous at \( w_0 \). It also follows that
\[
g(w) - g(w_0) = (g'(w_0) + \Phi(w))(w - w_0)
\]
for all \( w \in W \). Now choose \( \delta > 0 \) so that \( f \) is defined for all \( z \) in the \( \delta \) neighborhood of \( z_0 \) and \( f(z) \in W \) whenever \( z \) is in this neighborhood (such a \( \delta \) exists because \( f \) is continuous at \( z_0 \)). Call this neighborhood \( U \). We then have that
\[
g(f(z)) - g(f(z_0)) = (g'(f(z_0)) + \Phi(f(z)))(f(z) - f(z_0))
\]
for all $z \in U$. Hence we have

$$(g \circ f)'(z_0) = \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0}$$

$$= \lim_{z \to z_0} (g'(f(z_0)) + \Phi(f(z))) \frac{f(z) - f(z_0)}{z - z_0}$$

$$= g'(f(z_0))f'(z_0).$$