Implicit Differentiation
Mathematics 11: Lecture 19

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Example

Let $y = x^{\frac{2}{3}}$.

Then $y^3 = x^2$.

Differentiating both sides of this expression with respect to $x$ gives us $\frac{d}{dx} y^3 = \frac{d}{dx} x^2$.

Treating $y$ as a function of $x$, and remembering the chain rule, we have $3y^2 \frac{dy}{dx} = 2x$.

Hence $\frac{dy}{dx} = \frac{2x}{3y^2}$.

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- Treating $y$ as a function of $x$, and remembering the chain rule, we have
  \[ 3y^2 \frac{dy}{dx} = 2x, \]
- Hence
  \[ \frac{dy}{dx} = \frac{2x}{3y^2} = \frac{2}{3}xy^{-2}. \]
Example (cont’d)

- Now $y = x^{\frac{2}{3}}$, so

$$y^{-2} = \left(x^{\frac{2}{3}}\right)^{-2} = x^{-\frac{4}{3}}.$$
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▶ Thus

$$\frac{dy}{dx} = \frac{2}{3} xx^{-\frac{4}{3}} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}.$$ 

Note: this shows that $\frac{d}{dx} x^{\frac{2}{3}} = \frac{2}{3} x^{-\frac{1}{3}}$, which agrees with our previous rule for differentiating $x^n$ when $n$ is an integer.
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- Now \( y = x^{2/3} \), so
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  \]

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  \frac{dy}{dx} = \frac{2}{3}x^{-4/3} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.
  \]

- Note: this shows that
  \[
  \frac{d}{dx} x^{2/3} = \frac{2}{3}x^{-1/3} = \frac{2}{3}x^{2/3-1},
  \]
  which agrees with our previous rule for differentiating \( x^n \) when \( n \) is an integer.
If $n$ is a rational number, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$
Proof

▶ Let \( n = \frac{p}{q} \), where \( p \) and \( q \) are integers. If \( y = x^{\frac{p}{q}} \), then \( y^q = x^p \).

Differentiating both sides with respect to \( x \), we have

\[
qy^{q-1}\frac{dy}{dx} - 1 = px^{p-1}
\]

Thus

\[
\frac{dy}{dx} = \frac{p}{q} x^{\frac{p}{q} - 1} y^{\frac{1}{q} - \frac{1}{1}} = \frac{p}{q} x^{\frac{p}{q} - 1} x^{\frac{q}{q} - \frac{p}{q}} = \frac{p}{q} x^{\frac{p}{q}} - 1 = nx^{n - 1}.
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Proof

- Let $n = \frac{p}{q}$, where $p$ and $q$ are integers. If $y = x^{\frac{p}{q}}$, then $y^q = x^p$.
- Differentiating both sides with respect to $x$, we have

\[ qy^{q-1} \frac{dy}{dx} = px^{p-1}. \]

- Thus

\[ \frac{dy}{dx} = \frac{p}{q} x^{p-1} \frac{1}{y^{1-q}} \]

\[ = \frac{p}{q} x^{p-1} x^{\frac{p}{q} - p} \frac{1}{q} \]

\[ = \frac{p}{q} x^{\frac{p}{q} - 1} = nx^{n-1}. \]
If \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \), then

\[
f'(x) = -\frac{1}{2}(1 + x^2)^{-\frac{3}{2}}(2x) = -\frac{x}{(1 + x^2)^{\frac{3}{2}}}.
\]
Implicit differentiation

The technique used to find the derivative of a rational power is useful in finding the slope of a curve defined by an equation, but not explicitly expressed as the graph of a function. We call this *implicit differentiation.*
Example

Consider the equation

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Suppose we wish to find the slope of $C$ at a point, say, the point $(3, 4)$. 

Note: such an explicit expression would work for only some points on $C$ since, overall, $C$ is not the graph of a function.
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- The graph of this equation is the circle $C$ of radius 5 centered at the origin.

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- One way to proceed would be to solve for $y$ explicitly in terms of $x$ and then differentiate the resulting expression.
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The graph of this equation is the circle \( C \) of radius 5 centered at the origin.

Suppose we wish to find the slope of \( C \) at a point, say, the point \((3, 4)\).

One way to proceed would be to solve for \( y \) explicitly in terms of \( x \) and then differentiate the resulting expression.

Note: such an explicit expression would work for only some points on \( C \) since, overall, \( C \) is not the graph of a function.
Example (cont’d)

- Graph of $x^2 + y^2 = 25$ with tangent line at $(3, 4)$:
Example (cont’d)

▶ Another approach: differentiate both sides of the expression,

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25),
\]

to obtain

\[
2x + 2y \frac{dy}{dx} = 0.
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Note that since we are differentiating with respect to \( x \), we treated \( y \) as a function of \( x \) and used the chain rule to differentiate \( y^2 \).
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Note that since we are differentiating with respect to \( x \), we treated \( y \) as a function of \( x \) and used the chain rule to differentiate \( y^2 \).

Now we may solve for \( \frac{dy}{dx} \), obtaining

\[ \frac{dy}{dx} = -\frac{x}{y}. \]
Example (cont’d)

- Evaluating at our point of interest, we have

\[
\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.
\]

So the equation of the line tangent to \(C\) at \((3,4)\) is

\[
y = -\frac{3}{4}(x-3) + 4.
\]

Note: our expression for \(\frac{dy}{dx}\) is valid only when \(y \neq 0\).

The lines tangent to \(C\) at the two points where \(y = 0\), namely, \((-5,0)\) and \((5,0)\) are vertical. Hence we should not expect to find derivatives at those points.
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Example

Consider the curve $C$ with equation

$$x^4 + y^5 + 4xy^2 = 25.$$
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Differentiating both sides of this expression,

$$\frac{d}{dx}(x^4 + y^5 + 4xy^2) = \frac{d}{dx}(25),$$

gives us

$$4x^3 + 5y^4 \frac{dy}{dx} + 8xy \frac{dy}{dx} + 4y^2 = 0.$$
Example (cont’d)

- Solving for \( \frac{dy}{dx} \), we have

\[
\frac{dy}{dx} = -\frac{4x^3 + 4y^2}{5y^4 + 8xy}.
\]
Example (cont’d)

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$$\frac{dy}{dx} = -\frac{4x^3 + 4y^2}{5y^4 + 8xy}.$$  

▶ Now $(2, 1)$ is a point on $C$ and

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = -\left(\frac{32 + 4}{5 + 16}\right) = -\left(\frac{36}{21}\right) = -\frac{12}{7}.$$
Example (cont’d)

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$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = -\frac{32 + 4}{5 + 16} = -\frac{36}{21} = -\frac{12}{7}.$$

- Hence the equation of the line tangent to $C$ at $(2, 1)$ is

$$y = -\frac{12}{7}(x - 2) + 1.$$
Example (cont’d)

Graph of $x^4 + y^5 + 4xy^2 = 25$ with tangent line at $(2, 1)$: