An Analysis of Katsuura's Continuous Nowhere Differentiable Function

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1 Introduction

Explicit examples of continuous nowhere differentiable functions have been known since the 19th century. In 1872, for example, Karl Weierstrass, in a lecture before the Royal Academy of Science in Berlin, demonstrated that the function

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

is continuous but nowhere differentiable whenever $0 < a < 1$, $ab > 1 + 3\pi/2$, and $b > 1$ is an odd integer. Weierstrass' example was published in 1875 by Paul du Bois-Reymond. Because his was the first published example of a continuous nowhere differentiable function, Weierstrass is generally given credit for being the first to give such a construction. However examples of continuous nowhere differentiable functions had already been developed. Perhaps the first such construction was given by Bernard Bolzano, circa 1830. Bolzano's example is especially interesting in our present setting since it does not make use infinite series; his construction is essentially geometric. A fascinating account of the history of continuous and nowhere differentiable functions can be found in the master's thesis of Thim [9].

In 1991, Katsuura [7] published an example of a continuous nowhere differentiable function based on a fixed-point method. Let $X = [0,1]^2$ denote the unit square of the plane and let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X. $\mathcal{K}(X)$ is a complete metric space in the Hausdorff metric (see, for example, Theorem 2.4.4 of Edgar [3]). Let A_1, A_2, and A_3 be affine contractions on X given by the following rules: for $(x, y) \in X$, let

$$A_1(x,y) = \left(\frac{x}{3}, \frac{2y}{3}\right), \qquad A_2(x,y) = \left(\frac{2-x}{3}, \frac{1+y}{3}\right),$$
$$A_3(x,y) = \left(\frac{2+x}{3}, \frac{1+2y}{3}\right).$$

These maps are illustrated in Figure 1.

Given $F \in \mathcal{K}(X)$, let

$$A(F) = A_1(F) \cup A_2(F) \cup A_3(F).$$

Let $G_0 = \{(x,x) : x \in [0,1]\}$ and, for $k \geq 1$, let

$$G_k = A_1(G_{k-1}) \cup A_2(G_{k-1}) \cup A_3(G_{k-1}).$$

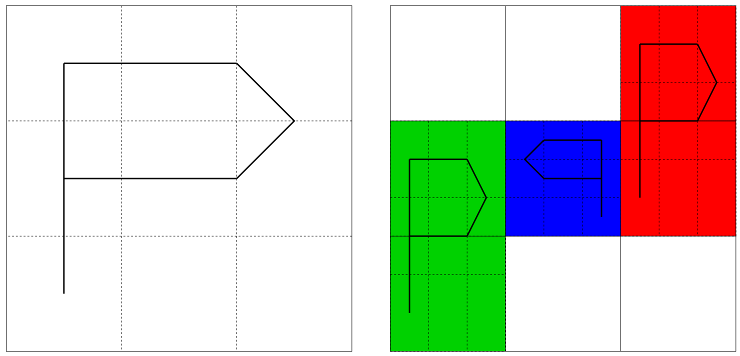

Figure 1: A_1 maps the unit square onto the green region by contraction; A_2 maps the unit square onto the blue region by contraction and reflection; and A_3 maps the unit square onto the red region by contraction.

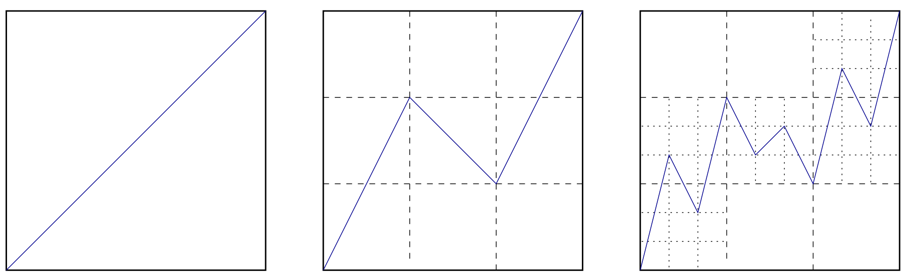

Figure 2: The functions f_0, f_1, and f_2 respectively.

Katsuura showed that A is a contraction map on $\mathcal{K}(X)$. As such, there is a unique fixed point G of A in $\mathcal{K}(X)$, and $G_k \to G$ in the Hausdorff metric. For each $k \geq 0$, G_k is the graph of a continuous function, which we will denote by f_k; see Figure 1. The set G is the graph of a function f, which is Katsuura's function. Katsuura showed that the sequence of functions $\{f_n\}$ converges uniformly to f and that f is continuous but nowhere differentiable. Katsuura's function is an example of self-affine dust and is closely related to the class of Kiesswetter's curves; see, for example, [3], p. 200ff.

Chuang and Lewis [2] studied the increments of Katsuura's function and showed that if x is chosen uniformly from the interval $[0, 1)$, then $\ln|f(x + h_k) - f(x)|$, suitably normalized, converges in measure to a standard normal random variable as $h_k \to 0^+$ along either $h_k = 1/3^k$ or $h_k =$

$2/3^k$, $k \to +\infty$. In part this result was anticipated in the literature, for probabilistic properties of continuous nowhere differentiable functions had already been investigated by Kono [8] and Gamkrelidze [5] for Takagi's function and by Gamkrelidze [4] for Weierstrass' function. There are two main themes in the present paper, both of which are motivated by the investigation Chuang and Lewis.

The chief shortcoming in the result Chuang and Lewis is the limitation on the approach of h_k to 0 along the special subsequences. The underlying issue is this: given $h > 0$ and x chosen uniformly from $[0, 1)$, is the random variable $\ln |f(x + h) - f(x)|$ proper? Stated another way, does the set

$$\{x \in [0, 1 - h) : f(x + h) - f(x) = 0\}$$

have Lebesgue measure 0? In Section 4 we answer this question in the affirmative provided that $h > 0$ is a triadic rational number, that is, $h = j/3^k$ for some positive integers j and k; see Theorem 4.1. [1] A crucial step in the proof of this result is a series representation of Katsuura's function which we develop in Section 3; see Lemma 3.1.

The second main theme of this paper is focused on quantifying the continuity of f. In Theorem 5.1 of Section 5 we give an explicit exponential rate for the uniform modulus of continuity of f and in Theorem 6.1 of Section 6 we give, among other things, an explicit exponential rate for the local modulus of continuity of f. We show that f has the same local behavior at almost all of the points in $[0, 1]$ and that this local behavior is quite different from the global or uniform behavior of f. In this way Katsuura's function acts very much like the path of a Brownian motion. The uniform behavior of a Brownian path is given by Lévy's modulus of continuity while local behavior is given by the law of the iterated logarithm; see, for example, Theorems 9.23 and 9.25 of [6].

2 Preliminary definitions and results

For $k \geq 0$, let $D_k = \{j/3^k : j \in \mathbb{Z}\}$ and let $D = \cup_{k \geq 1} D_k$, the so-called triadic rational numbers. Throughout let $t \in [0, 1)$ and let $t = .t_1 t_2 t_3 \ldots$ be the unique ternary expansion of t.[2]

[1] It is still open as to whether or not the set in question has measure 0 for *any* $h > 0$.

[2] Unique in the sense that $t - .t_1 t_2 \ldots t_k < 3^{-k}$. In this way we eliminate ternary expansions ending in repeating 2's.

Let $St = 3t \pmod 1$. Note that S is simply the shift operator on the ternary expansion of t, that is,

$$S(.t_1t_2t_3\cdots) = .t_2t_3\cdots$$

We will let S^k denote the k-fold self-composition of S.

For $k \geq 1$, let T^k be the *truncation operator* $T^kt = 3^{-k}\lfloor 3^kt\rfloor$. Effectively T^k chops off all of the digits in the ternary expansion of a number after the kth place; thus,

$$T^k(.t_1t_2\dots t_kt_{k+1}\dots) = .t_1t_2\dots t_k000\dots$$

Finally let $\sigma_0(t) = 0$ and, for $k \geq 1$, let

$$\sigma_k(t) = \sigma_k(.t_1t_2t_3\dots) = \#\{i : t_i = 1, 1 \leq i \leq k\}.$$

Thus $\sigma_k(t)$ merely counts the number of times 1 appears in the first k terms of the ternary expansion of t. Our next result is well known, but we include it for completeness.

2.1 Theorem. *For each $k \geq 1$, $\sigma_k : [0,1) \to \mathbb{R}$ is a binomial random variable with parameters k and $1/3$ with respect to Lebesgue measure.*

Proof. Our proof is a minor modification of the analogous result for dyadic expansions; see, for example, pages 3 through 5 of Billingsley [1]. Let $\mathbf{m}$ denote Lebesgue measure.

Let $u_1, u_2, \dots, u_k$ be elements from $\{0,1,2\}$. Since

$$\{x \in [0,1) : T^kx = .u_1u_2\dots u_k\} = [.u_1u_2\dots u_k, .u_1u_2\dots u_k + 1/3^k),$$

we may conclude that

$$\mathbf{m}\{x \in [0,1) : T^kx = .u_1u_2\dots u_k\} = \frac{1}{3^k}.$$

Since there are $\binom{k}{i}2^{k-i}$ arrangements of $u_1, u_2, \dots, u_k$ containing exactly k 1's, it follows that

$$\mathbf{m}\{x \in [0,1) : \sigma_k(x) = i\} = \binom{k}{i}2^{k-i}\frac{1}{3^k} = \binom{k}{i}(1/3)^i(2/3)^{k-i},$$

as was to be shown. $\square$

For $t \in [0, 1)$, let

$$p_k(t) = (2/3)^{k-\sigma_k(t)}(-1/3)^{\sigma_k(t)} = \frac{2^k(-2)^{-\sigma_k(t)}}{3^k}. \tag{1}$$

2.1 Lemma. *Let $k \geq 1$. If α and β are consecutive elements of $D_k \cap [0, 1)$, then*

(1) $\sigma_k(\beta) = \sigma_k(\alpha) \pm 1$ *and*

(2) $|p_k(\beta)| = 2^{\pm 1}|p_k(\alpha)|$.

Proof. We will prove (1) by induction on k. The claim is certainly true if $k = 1$; the elements of $D_1 \cap [0, 1)$ are $.0$, $.1$, and $.2$, and the corresponding values of σ_1 are 0, 1, and 0.

Let us suppose that the result is true for some $k \geq 1$ and let

$$\alpha = .\alpha_1 \ldots \alpha_k \quad \text{and} \quad \beta = .\beta_1 \ldots \beta_k$$

be consecutive elements of $D_k \cap [0, 1)$ with $\alpha < \beta$; thus, we may assume that

$$\sigma_k(\beta) = \sigma_k(\alpha) \pm 1. \tag{2}$$

Then

$$.\alpha_1 \ldots \alpha_k 0, \quad .\alpha_1 \ldots \alpha_k 1, \quad .\alpha_1 \ldots \alpha_k 2, \quad .\beta_1 \ldots \beta_k 0$$

will be consecutive elements in $D_{k+1} \cap [0, 1)$, and the corresponding values of σ_{k+1} for these numbers will be $\sigma_k(\alpha)$, $\sigma_k(\alpha) + 1$, $\sigma_k(\alpha)$, and $\sigma_k(\beta) = \sigma_k(\alpha) \pm 1$, by (2). By inspection we find that the claim is true for these consecutive elements of $D_{k+1} \cap [0, 1)$ as well.

The proof of (2) follows trivially from (1) and the definition of p_k. $\square$

We will give an alternate description of the sequence of functions $\{f_k\}$. Let $f_0(x) = x$ be the identity function. Thereafter let

$$f_k(t) = \begin{cases} \frac{2}{3} f_{k-1}(St) & \lfloor 3t \rfloor = 0 \\ -\frac{1}{3} f_{k-1}(St) + \frac{2}{3} & \lfloor 3t \rfloor = 1 \\ \frac{2}{3} f_{k-1}(St) + \frac{1}{3} & \lfloor 3t \rfloor = 2 \end{cases} \tag{3}$$

We can summarize this in the compact formula:

$$f_k(t) = p_1(t) f_{k-1}(St) + f_1(Tt). \tag{4}$$

Formula (4) can be easily extended by induction.

2.2 Theorem. *For each* $k \geq 1$

$$f_k(t) = p_j(t) f_{k-j}(S^j t) + f_j(T^j t).$$

for all $t \in [0, 1)$ *and for each* $1 \leq j \leq k$.

Proof. The statement is true in the case $k = 1$ and $j = 1$, by the very definition of f_1. Now let us assume that the claim is true up to index $k - 1$ and let us consider the validity of this claim for index k.

Note that equation (4) establishes the base case, $j = 1$. Now let us assume that this result is true up to an index $1 \leq j < k$. Then we find that

$$\begin{aligned}
f_k(t) &= p_j(t) f_{k-j}(S^j t) + f_j(T^j t) \\
&= p_j(t) \left\{ p_1(S^j t) f_{k-j-1}(S^{j+1} t) + f_1(T S^j t) \right\} + f_j(T^j t) \qquad \text{(by eq. (4))} \\
&= p_j(t) p_1(S^j t) f_{k-(j+1)}(S^{j+1} t) + p_j(t) f_1(T S^j t) + f_j(T^j t)
\end{aligned}$$

Observe that $p_j(t) p_1(S^j t) = p_{j+1}(t)$. By our induction hypothesis,

$$f_{j+1}(t) = p_j(t) f_1(S^j t) + f_j(T^j t);$$

thus, upon substituting $T^{j+1} t$ for t in the above, we obtain

$$f_{j+1}(T^{j+1} t) = p_j(T^{j+1} t) f_1(S^j T^{j+1} t) + f_j(T^j T^{j+1} t).$$

However, $p_j(T^{j+1} t) = p_j(t)$, $S^j T^{j+1} t = T S^j t$, and $T^j T^{j+1} t = T^j t$. Consequently,

$$p_j(t) f_1(T S^j t) + f_j(T^j t) = f_{j+1}(T^{j+1} t).$$

Summarizing our findings, we have

$$f_k(t) = p_{j+1}(t) f_{k-(j+1)}(S^{j+1} t) + f_{j+1}(T^{j+1} t),$$

as was to be shown. □

2.3 Theorem. *For each* $j \geq 1$,

$$f(t) = p_j(t) f(S^j t) + f(T^j t). \tag{5}$$

Proof. Recall that Katsuura's function is the uniform limit of the sequence $\{f_k\}$ on the unit interval. Thus by taking limits as k tends to infinity in the formula of Theorem 2.2, we obtain

$$f(t) = p_j(t) f(S^j t) + f_j(T^j t). \tag{6}$$

We are left to show that $f_j(T^j t) = f(T^j t)$. Notice that if $u \in D_j \cap [0,1)$, then $T^j u = u$, $S^j u = 0$, and equation (6) becomes

$$f(u) = f_j(u).$$

In other words, f agrees with f_j on $D_j \cap [0,1)$. [3] In particular, since $T^j t \in D_j \cap [0,1)$, $f(T^j t) = f_j(T^j t)$, which finishes our proof. $\square$

We finish this section with two simple consequences of Theorem 2.3.

2.1 Corollary. *Let $k \geq 1$ and let α, β be consecutive elements of D_k with $\alpha < \beta$. Then $f(\beta) - f(\alpha) = p_k(\alpha)$.*

Proof. For $x \in (\alpha, \beta)$, $T^k x = \alpha$ and, by Theorem 2.3, $f(x) = p_k(\alpha) f(S^k x) + f(\alpha)$ As $x \to \beta^-$, $S^k x \to 1$ and, since f is continuous, this implies

$$f(\beta) = p_k(\alpha) f(1) + f(\alpha) = p_k(\alpha) + f(\alpha),$$

as was to be shown. $\square$

2.2 Corollary. *Let $k \geq 1$ and let α, β be consecutive elements of D_k with $\alpha < \beta$. If $\alpha \leq x \leq \beta$, then $f(x)$ is between $f(\alpha)$ and $f(\beta)$.*

Proof. The claim is trivial if $x = \alpha$ or $x = \beta$; thus, we may assume that $\alpha < x < \beta$. In this case, note that $T^k x = \alpha$ and $p_k(x) = p_k(\alpha)$; let $u = S^k x$. Then, by Theorem 2.3,

$$f(x) = p_k(\alpha) f(u) + f(\alpha).$$

Recall that $0 \leq f(u) \leq 1$ and, by Corollary 2.1, $f(\beta) - f(\alpha) = p_k(\alpha)$. Thus if $p_k(\alpha) > 0$, then

$$f(\alpha) \leq f(x) \leq f(\alpha) + p_k(a) = f(\beta).$$

If $p_k(\alpha) < 0$, then

$$f(\beta) = f(\alpha) + p_k(\alpha) \leq f(x) \leq f(\alpha).$$

In either case, the claim has been established. $\square$

[3] This assertion is quite clear from Katsuura's geometric definition, but it is not difficult to see on purely analytic grounds as well.

3 A representation formula

In this section we will present a series representation of the Katsuura function.

3.1 Lemma. *For* $t \in [0, 1)$,

$$f(t) = \sum_{k=0}^{\infty} p_k(t) f(.t_{k+1}) \tag{7}$$

Note please that $f(.t_{k+1})$ will be 0, $2/3$ or $1/3$ depending upon whether $t_{k+1} = 0, 1$ or 2 respectively.

Proof. Let $t = .t_1 t_2 t_3 \cdots \in [0, 1)$, and recall

$$T^k t = .t_1 t_2 \ldots t_k 00 \ldots$$

Since $f(t) = \lim_{k\to\infty} f(T_k t)$, we have

$$\begin{aligned} f(t) &= \lim_{k\to\infty} f(T_1 t) + \big(f(T_2 t) - f(T_1 t)\big) + \cdots + \big(f(T_k t) - f(T_{k-1} t)\big) \\ &= f(T_1 t) + \sum_{k=1}^{\infty} \big(f(T_{k+1} t) - f(T_k t)\big) \end{aligned}$$

However, by Theorem 2.3 we see that

$$f(T_{k+1} t) - f(T_k t) = p_k(T_{k+1} t) f(S_k T_{k+1} t) = p_k(t) f(.t_{k+1}).$$

Recalling that $p_0(t) = 1$ for all t, we have

$$f(t) = \sum_{k=0}^{\infty} p_k(t) f(.t_{k+1}),$$

as was to be shown. $\square$

By Lemma 3.1, we may write

$$f(t) = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k H_k(t),$$

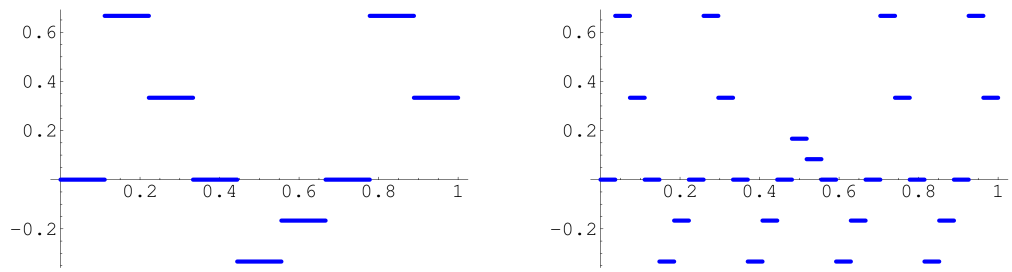

Figure 3: The graphs of H_2 and H_3 respectively.

where

$$H_k(t) = \left(-\frac{1}{2}\right)^{\sigma_k(t)} f(.t_{k+1}).$$

The graphs of H_2 and H_3 are given in Figure 3. Observe that as k increases, the functions $H_k(t)$ increase in "frequency". Thus we have a representation of Katsuura's function as a sum of terms decreasing in amplitude but increasing in frequency as k increases. In this sense, our representation bears some resemblance to the example of Weierstrass.

4 Concerning the increments

Let $\mathbf{m}$ denote Lebesgue measure. Here is the main result of this section.

4.1 Theorem. *If $h \in (0,1)$ is a triadic rational number, then*

$$\mathbf{m}\{t \in [0, 1-h) : f(t+h) - f(t) = 0\} = 0.$$

We begin with a lemma that makes direct use of our representation formula (7).

4.1 Lemma. *Let $\alpha, t \in D_k \cap [0,1)$, $k \geq 1$. If $p_k(\alpha) = p_k(t)$, then $f(\alpha) \neq f(t)$.*

Proof. Since $\alpha, t \in D_k \cap [0,1)$ for some $k \geq 1$, we have, by Lemma 3.1,

$$f(t) = \sum_{j=0}^{k-1} p_j(t) f(.t_{j+1})$$

$$f(\alpha) = \sum_{j=0}^{k-1} p_j(\alpha) f(.\alpha_{j+1})$$

and therefore

$$f(t) - f(\alpha) = \sum_{j=0}^{k-1} \big(p_j(t) f(.t_{j+1}) - p_j(\alpha) f(.\alpha_{j+1})\big).$$

Let $\nu = \max\{i : \alpha_i \neq t_i\}$. The condition $p_k(\alpha) = p_k(t)$ is equivalent to the condition that α and t have the same number of 1's in their ternary expansions. Since the ternary digits of α and t agree after index ν, it follows that

$$p_j(\alpha) f(.\alpha_{j+1}) = p_j(t) f(.t_{j+1}) \quad \text{for } j \geq \nu,$$

and we may conclude that

$$f(t) - f(\alpha) = \sum_{j=0}^{\nu-1} \big(p_j(t) f(.t_{j+1}) - p_j(\alpha) f(.\alpha_{j+1})\big).$$

Recalling the definition of p_k, we may write

$$3^\nu \big(f(t) - f(\alpha)\big) = \sum_{j=0}^{\nu-1} 3^{\nu-1-j} 2^j \big((-2)^{-\sigma_j(t)} 3 f(.t_{j+1}) - (-2)^{-\sigma_j(\alpha)} 3 f(.\alpha_{j+1})\big)$$

Observe that every term save the last in the sum on the right-hand side is a multiple of 3; thus,

$$3^\nu \big(f(t) - f(\alpha)\big) = 2^{\nu-1} \big((-2)^{-\sigma_{\nu-1}(t)} 3 f(.t_\nu) - (-2)^{-\sigma_{\nu-1}(\alpha)} 3 f(.\alpha_\nu)\big) \pmod 3$$

Since $2 = -1 \pmod 3$ and $-2 = 1 \pmod 3$, we may simply write

$$3^\nu \big(f(t) - f(\alpha)\big) = (-1)^{\nu-1} \big(3 f(.t_\nu) - 3 f(.\alpha_\nu)\big) \pmod 3$$

By definition, $t_\nu \neq \alpha_\nu$, and an elementary listing of all of the cases shows that

$$3 f(.t_\nu) - 3 f(.\alpha_\nu)\big) \neq 0 \pmod 3$$

which shows that $f(t) \neq f(\alpha)$, as was to be shown. $\square$

We are now prepared to prove Theorem 4.1.

Proof of Theorem 4.1. Let $h \in (0,1)$ be a triadic rational and let $t \in [0, 1-h)$. Let us assume that $h = .h_1h_2 \dots h_k000\dots$ for some $k \geq 1$ and let $t = t_1t_2 \dots t_kt_{k+1}\dots$. Then

$$t + h = .\alpha_1\alpha_2 \dots \alpha_k t_{k+1} \dots$$

According to Theorem 2.3, we have

$$\begin{aligned} f(t+h) &= p_k(.\alpha_1\alpha_2 \dots \alpha_k) f(S^k t) + f(.\alpha_1\alpha_2 \dots \alpha_k) \\ f(t) &= p_k(.t_1t_2 \dots t_k) f(S^k t) + f(t_1t_2 \dots t_k) \end{aligned}$$

Thus

$$f(t+h) - f(t) = \Delta p f(S^k t) + \Delta f,$$

where

$$\begin{aligned} \Delta p &= p_k(.\alpha_1\alpha_2 \dots \alpha_k) - p_k(.t_1t_2 \dots t_k) \\ \Delta f &= f(.\alpha_1\alpha_2 \dots \alpha_k) - f(.t_1t_2 \dots t_k) \end{aligned}$$

If we let E denote the set of triadic rationals $.t_1t_2 \dots t_k$ such that $t + h < 1$, then

$$\{t \in [0, 1-h) : f(t+h) - f(t) = 0\} = \bigcup_{.t_1t_2\dots t_k \in E} \{u \in [0,1) : \Delta p f(u) + \Delta f = 0\}$$

Since there are only finitely many elements in E, it is enough to show that

$$\mathbf{m}\{u \in [0,1) : \Delta p f(u) + \Delta f = 0\} = 0$$

for each $.t_1t_2 \dots t_k \in E$. There are two cases to consider.

(1) If $\Delta f \neq 0$, then we must consider the equation

$$\Delta p f(u) = -\Delta f$$

If $\Delta p = 0$, then there are no solutions to this equation, and the set in question has measure 0. If $\Delta p \neq 0$, then we seek the measure of the level set

$$\{u \in [0,1) : f(u) = -\Delta f / \Delta p\}$$

As was shown in Chuang and Lewis [2], this level set has measure 0.

(2) If $\Delta f = 0$, then we know from Theorem 4.1 that $\Delta p \neq 0$ and thus we seek the measure of the level set

$$\{u \in [0,1) : f(u) = 0\},$$

which has only the trivial solution hence measure 0.

This completes our proof. $\square$

5 The uniform modulus of continuity

It is evident that Katsuura's function f fluctuates throughout the interval $[0,1]$. This fluctuation can be quantified: for $0 < h < 1$, the function

$$\omega(h) = \sup_{\substack{s,t\in[0,1]\\|s-t|\le h}} |f(t) - f(s)|,$$

called the *uniform modulus of continuity of* f, is a measure of the oscillation of f throughout the interval $[0,1]$. Let

$$\Gamma = \frac{\log_2(3/2)}{\log_2(3)} = 1 - \log_3(2) = 0.36907\ldots \tag{8}$$

5.1 Theorem. *If $h \in (0,1)$, then $\frac{2}{3}h^\Gamma \le \omega(h) \le \frac{3}{2}h^\Gamma$. In particular,*

$$\lim_{h\to 0^+} \frac{\log_2 \omega(h)^{-1}}{\log_2(h^{-1})} = \Gamma$$

Proof. The proof is composed of upper and lower bound arguments. The lower bound argument is the easier of the two, and we will begin there.

Let $1/3^k \le h < 1/3^{k-1}$, $k \ge 1$. Since $h \mapsto \omega(h)$ is increasing in h,

$$\omega(h) \ge \omega(1/3^k) \ge |f(1/3^k) - f(0)| = f(1/3^k) = (2/3)^k > \frac{2}{3}h^\Gamma,$$

which is the lower bound.

In the argument for the upper bound we will make use of the observation that f has an odd symmetry about the point $(1/2, 1/2)$. This may be stated as

$$f(1-y) + f(y) = 1, \quad \text{for all } y \in [0,1]$$

This symmetry is exhibited on smaller scales throughout the interval $[0,1]$. Thus suppose that x_j and x_{j+1} are consecutive elements of D_k. Then for $y \in [x_j, x_{j+1}]$ we have

$$f(x_{j+1} - y) + f(x_j + y) = f(x_{j+1}) + f(x_j). \tag{9}$$

Let $s, t \in D_k \cap [0,1)$ for some $k \ge 1$ with $s < t$ and let $h = t - s = j/3^k$. Let us assume that $3^\ell \le j < 3^{\ell+1}$ and therefore $3^{\ell-k} \le h < 3^{\ell+1-k}$. This implies that

$$h = .00\ldots 0 h_{k-\ell} h_{k-\ell+1} \ldots h_k, \quad h_{k-\ell} \ne 0,$$

Thus

$$f(h) = (2/3)^{k-\ell-1} f(.h_{k-\ell} h_{k-\ell+1} \dots h_k)$$

Since f is bounded by 1, we may conclude that

$$f(h) \leq (2/3)^{k-\ell-1} = (2/3)^{k-\ell} \frac{3}{2} = \frac{3}{2} h^{\Gamma}. \tag{10}$$

This is our fundamental starting point.

Let us consider various cases for the relationship between s and t.

(1) Suppose that $0 \leq s < 1/3$ and $1/3 \leq t < 2/3$. Then

$$\begin{aligned} f(t) - f(s) &= (f(t) - f(1/3)) + (f(1/3) - f(s)) \\ &= -(f(1/3) - f(t)) + (f(1/3) - f(s)). \end{aligned}$$

In this case t and $1/3 = .1$ share the same first ternary digit; thus,

$$0 \leq (f(1/3) - f(t)) = -1/3(f(0) - f(St)) = \frac{1}{3} f(St).$$

But $|St| = 3|t - 1/3| \leq 3h$ and therefore

$$0 \leq (f(1/3) - f(t)) \leq \frac{1}{3}\frac{3}{2}|3h|^{\Gamma}$$

Since $3^{\gamma} = 3/2$, it follows that

$$0 \leq (f(1/3) - f(t)) \leq \frac{3}{2} h^{\Gamma}.$$

Likewise, by our symmetry formula (9),

$$0 \leq f(1/3) - f(s) = f(1/3 - s) \leq \frac{3}{2}|1/3 - s|^{\Gamma} \leq \frac{3}{2} h^{\Gamma}.$$

In summary, $|f(t) - f(s)| \leq \frac{3}{2}|t - s|^{\Gamma}$ for $s \in [0, 1/3) \cap D_k$ and $t \in [1/3, 2/3) \cap D_k$.

(2) If $s \in [1/3, 2/3) \cap D_k$ and $t \in [2/3, 1) \cap D_k$, then, arguing as above, we can show that

$$|f(t) - f(s)| \leq \frac{3}{2}|t - s|^{\Gamma}$$

in this case as well.

(3) Suppose that $s \in [0, 1/3) \cap D_k$ and $t \in [2/3, 1) \cap D_k$. Then

$$|f(t) - f(s)| \leq 1 = \frac{3}{2}\frac{2}{3} = \frac{3}{2}\left(\frac{1}{3}\right)^{\Gamma} \leq \frac{3}{2}h^{\Gamma}.$$

(4) Finally let us suppose that s and t are in one of the subintervals: $[0, 1/3) \cap D_k$, $[1/3, 2/3) \cap D_k$, or $[2/3, 1) \cap D_k$. In this case the ternary expansions of s and t share at least the first digit. Let us suppose that s and t share the same first p digits, $p \geq 1$. It follows that $S^p t$ and $S^p s$ must conform to one of our previous cases (1)-(3). Since $|S^p t - S^p s| = 3^p |t - s| = 3^p h$,

$$|f(t) - f(s)| = (2/3)^p |f(S^p t) - f(S^p s)| \leq \frac{3}{2}(2/3)^p (3^p h)^{\Gamma} = \frac{3}{2}h^{\Gamma}.$$

Finally given $s, t \in [0, 1)$ we note that $T_k t$ and $T_k s$ are in $D_k \cap [0, 1)$; hence,

$$\begin{aligned} |f(t) - f(s)| &\leq |f(t) - f(T_k t)| + |f(T_k t) - f(T_k s)| + |f(T_k s) - f(s)| \\ &\leq |f(t) - f(T_k t)| + \frac{3}{2}|T_k t - T_k s|^{\Gamma} + |f(T_k s) - f(s)| \end{aligned}$$

By letting k increase without bound, we may conclude that

$$|f(t) - f(s)| \leq \frac{3}{2}|t - s|^{\Gamma}$$

This shows that $\omega(h) \leq \frac{3}{2}h^{\Gamma}$, which concludes the argument for the upper bound. □

6 The local modulus of continuity

In the previous section we studied the uniform continuity of f on $[0, 1]$; in this section we will study the continuity of f at a point. To this end, for $x \in [0, 1)$ and $h \in [0, 1/2)$, let

$$\omega(x, h) = \sup_{s,t \in [x-h, x+h] \cap [0,1]} |f(t) - f(s)|$$

This is the *modulus of continuity of f at x*. We will present three results concerning the behavior of the local modulus of continuity as $h \to 0^+$.

As we shall see, the typical value of $\omega(x,h)$ is of smaller order than the corresponding value of $\omega(h)$. In other words, it is the contributions of the local moduli from an exceptional set of points which influences the uniform modulus.

Let

$$\gamma = \Gamma + \frac{1}{3}\log_3(2) = 1 - \frac{2}{3}\log_3(2) = 0.57938\ldots \tag{11}$$

Our results on the local modulus of continuity of f are consequences of our next theorem.

6.1 Theorem. *There exist constants C_1 and C_2 such that*

$$C_1 \leq \log_2\left(\frac{h^\gamma}{\omega(x,h)}\right) - \left(\sigma_k(x) - \frac{k}{3}\right) \leq C_2$$

for all $x \in [0,1)$, where $k = k(h)$ is the smallest positive integer such that $2/3^k \leq h < 2/3^{k-1}$.

This theorem demonstrates that $\omega(x,h)$ is closely related to $\sigma_k(x)$; consequently, results about $\omega(x,h)$ can be lifted from results about $\sigma_k(x)$. In Theorem 2.1 it was shown that if x is chosen uniformly from $[0,1)$, then $\sigma_k(x)$ has a binomial distribution with parameters k and $1/3$. The following three results (the strong law of large numbers, the central limit theorem, and the law of the iterated logarithm) come from this observation:

(1) For almost all $x \in [0,1)$,

$$\lim_{k\to\infty} \frac{\sigma_k(x)}{k} = \frac{1}{3}.$$

(2) For any $a, b \in \mathbb{R}$, $a < b$,

$$\mathbf{m}\left\{x \in [0,1) : a \leq \frac{\sigma_k(x) - k/3}{\sqrt{(2/9)k}} \leq b\right\} \to \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du.$$

(3) For almost all $x \in [0,1)$,

$$\liminf_{k\to\infty} \frac{\sigma_k(x) - k/3}{\sqrt{(2/9)k\ln\ln(k)}} = -1 \quad \text{and} \quad \limsup_{k\to\infty} \frac{\sigma_k(x) - k/3}{\sqrt{(2/9)k\ln\ln(k)}} = 1.$$

The relationship between k and h stated in Theorem 6.1 implies that

$$D_1 \leq k - \log_3(2)\log_2(h^{-1}) \leq D_2. \tag{12}$$

Thus we can restate items (1), (2), and (3) above in the following form:

(1) For almost all $x \in [0,1)$,

$$\lim_{h\to 0^+} \frac{\log_2 \omega(x,h)^{-1}}{\log_2(h^{-1})} = \gamma.$$

This result should be compared with that of Theorem 5.1.

(2) Let

$$\varphi(h) = \sqrt{\frac{2}{9}\log_2(h^{-1})\log_3(2)}$$

For any $a, b \in \mathbb{R}$, $a < b$, as $h \to 0^+$,

$$\mathbf{m}\left\{x \in [0,1) : \frac{1}{\varphi(h)} \log_2\left(\frac{h^\gamma}{\omega(x,h)}\right) \in [a,b]\right\} \to \int_a^b e^{-u^2/2} du.$$

Thus, for example, as $h \to 0^+$, 68% of the points x in the interval $[0,1)$ satisfy

$$\log_2(h^{-\gamma}) - \varphi(h) \leq \log_2 \omega(x,h)^{-1} \leq \log_2(h^{-\gamma}) + \varphi(h).$$

(3) Let

$$\psi(h) = \sqrt{\frac{2}{9}\log_3(2)\log_2(1/h)\ln\ln\log_2(1/h)}.$$

Then, for almost all x in $[0,1)$,

$$\liminf_{h\to 0^+} \frac{\log_2 \omega(x,h)^{-1} - \log_2 h^{-\gamma}}{\psi(h)} = -1$$

and

$$\limsup_{h\to 0^+} \frac{\log_2 \omega(x,h)^{-1} - \log_2 h^{-\gamma}}{\psi(h)} = 1.$$

In particular, for any $\varepsilon > 0$, eventually

$$\log_2 h^{-\gamma} - (1+\varepsilon)\psi(h) \leq \log_2 \omega(x,h)^{-1} \leq \log_2 h^{-\gamma} + (1+\varepsilon)\psi(h)$$

while infinitely often

$$\log_2 \omega(x,h)^{-1} \geq \log_2 h^{-\gamma} + (1-\varepsilon)\psi(h)$$

and infinitely often

$$\log_2 \omega(x,h)^{-1} \leq \log_2 h^{-\gamma} - (1-\varepsilon)\psi(h)$$

We turn now to the proof of Theorem 6.1, and we begin with a lemma on bounding the increments of f.

6.1 Lemma. *Let a, $b \in D_k \cap [0,1)$ with $a < b$ and suppose that $b - a = \ell/3^k$; let $x \in [a,b]$. If $s, t \in [a,b]$, then*

$$|f(t) - f(s)| \leq \ell 2^\ell |p_k(x)|.$$

Proof. We will assume that $s < t$. Let $a = \alpha_0 < \alpha_1 < \cdots < \alpha_{\ell-1} < \alpha_\ell = b$ be the consecutive elements of D_k between a and b. First we will show that

$$|f(t) - f(s)| \leq \sum_{i=0}^{\ell-1} |p_k(\alpha_i)|. \tag{13}$$

Let us suppose that $s, t \in [\alpha_j, \alpha_{j+1}]$ for some $j \in \{0, 1, \ldots, \ell-1\}$. Then, by Corollary 2.2,

$$|f(t) - f(s)| \leq |f(\alpha_{j+1}) - f(\alpha_j)| = |p_k(\alpha_j)| \leq \sum_{i=0}^{\ell-1} |p_k(\alpha_i)|,$$

which verifies (13) in this case. Otherwise it must be that $s \in [\alpha_j, \alpha_{j+1})$ and $t \in (\alpha_m, \alpha_{m+1}]$ for some indices j and $m \in \{0, 1, \ldots, \ell-1\}$, with $j < m$. By telescoping and Corollary 2.2, we may write

$$\begin{aligned}
&|f(t) - f(s)| \\
&\quad \leq |f(t) - f(\alpha_m)| + |f(\alpha_m) - f(\alpha_{m-1})| + \cdots + |f(\alpha_{j+1}) - f(s)| \\
&\quad \leq \sum_{i=j}^{m} |f(\alpha_{i+1}) - f(\alpha_i)| = \sum_{i=j}^{m} |p_k(\alpha_i)| \\
&\quad \leq \sum_{i=0}^{\ell-1} |p_k(\alpha_i)|,
\end{aligned}$$

which, once again, verifies (13).

To finish the proof, observe that $x \in [a,b]$ implies that $p_k(x) = p_k(\alpha_j)$ for some $j \in \{0, 1, \ldots, \ell\}$, and, by Lemma 2.1,

$$|p_k(\alpha_i)| \le 2^\ell |p_k(x)|$$

for each $i \in \{0, 1, \ldots, \ell - 1\}$; thus,

$$|f(t) - f(s)| \le \sum_{i=0}^{\ell-1} |p_k(\alpha_i)| \le \ell 2^\ell |p_k(x)|,$$

as was to be shown. □

Proof of Theorem 6.1. Fix $x \in [0,1]$. Let α and β be elements of $D_k \cap [0,1]$ be chosen so that

$$[\alpha, \beta] \supset [x-h, x+h] \cap [0,1)$$

with $\beta - \alpha$ minimized. Then

$$\alpha > x - h - \frac{1}{3^k} \quad \text{and} \quad \beta < x + h + \frac{1}{3^k}$$

and $\beta - \alpha < 2h + 2/3^k < 14/3^k$, where we have the assumption that $h < 2/3^{k-1}$ to obtain the last inequality. It follows that $\beta - \alpha = \ell/3^k$, with $\ell \le 13$. By Lemma 6.1,

$$\omega(x,h) \le \sup_{s,t \in [\alpha,\beta]} |f(t) - f(s)| \le 13 \cdot 2^{13} |p_k(x)|,$$

which gives us an upper bound on the modulus at x.

To produce a corresponding lower bound, let $\alpha = .x_1 x_2 \ldots x_k$ and let $\beta = \alpha + 1/3^k$. Then $\alpha, \beta \in D_k \cap [0,1]$ and $\alpha, \beta \in [x-h, x+h] \cap [0,1]$. Thus, by Corollary 2.1,

$$\omega(x,h) \ge |f(\beta) - f(\alpha)| = |p_k(\alpha)| = |p_k(x)|.$$

In summary

$$|p_k(x)| \le \omega(x,h) \le 13 \cdot 2^{13} |p_k(x)|. \tag{14}$$

for each x.

By combining (1) and (12), we may assert that there exist constants E_1 and E_2 such that

$$E_1 \le \log_2 |p_k(x)| + \left\{ \sigma_k(x) - \frac{k}{3} \right\} + \log_2(h^{-\gamma}) \le E_2$$

This chain of inequalities in conjunction with (14) completes our proof. □

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