§2.1–Unions and Intersections Of Open and Closed Sets

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Open sets Closed sets

Outline

Example

Let $\{q_i, i \in \mathbb{N}\}$ be a listing of the rational numbers in $[0, 1]$. Let $A_i = (q_i - 1/4^i, q_i + 1/4^i)$ and let

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Is A open? Does A contain $[0, 1]$?

Theorem

An arbitrary (finite, countable, or uncountable) union of open sets is open.

Problem

Construct an example to show that an infinite intersection of open sets need not be open.

Theorem

The intersection of a finite number of open sets is open.

Theorem

An arbitrary (finite, countable, or uncountable) intersection of closed sets is closed.

Theorem

The union of a finite number of closed sets is closed.

Example (Cantor set)

Let $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$, and

$$C_2 = \underbrace{[0,1/9] \cup [2/9,1/3]}_{\frac{1}{3}C_1} \cup \underbrace{[2/3,7/9] \cup [8/9,1]}_{\frac{1}{3}C_1+\frac{2}{3}}$$

In general, let

$$C_n = \left(\frac{1}{3}C_{n-1}\right) \cup \left(\frac{1}{3}C_{n-1} + \frac{2}{3}\right).$$

Let $C = \cap_{n\geqslant 1} C_n$, called Cantor's middle-third set. Show that C is a closed set.