§1.2–Cuts

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Fall Term 2006

Outline

Elementary number systems

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Observe that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

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Theorem

No number in $\mathbb{Q}$ has a square equal to 2. In other words, $\sqrt{2} \notin \mathbb{Q}$.

R. Dedekind (1831 - 1916)

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Notation

We will write $A|B$ for the cut with left-hand part A and right-hand part B.

Problem

Is $A|B = \{r : r < 3\}|\{r : r > 3\}$ *a cut in* $\mathbb{Q}$*?*

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Definition

We call the cut $A|B$ a *rational cut* if there exists a rational number c such that

$$A|B = \{r : r < c\}|\{r : r \geq c\}$$

We will write c^* to denote this rational cut.

Problem

Is $A|B = \{r : r \leq 0 \text{ or } r^2 < 2\}|\{r : r > 0 \text{ and } r^2 \geq 2\}$ *a cut in* $\mathbb{Q}$*?*

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Problem

Is this cut a rational cut?

Definition

A *real number* is a cut in $\mathbb{Q}$. We will let $\mathbb{R}$ denote the class of cuts in $\mathbb{Q}$.

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Problem

Is every pair of cuts x and y comparable? That is, must one of the three statements be true:

$$x = y, \quad x < y \quad \textit{or} \quad y < x?$$

Definition

A cut $M \in \mathbb{R}$ is an *upper bound* for a set $S \subset \mathbb{R}$ provided that $s \leq M$ for each $s \in S$. Equivalently we say that S is *bounded above* by M.

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Definition

A cut M is a *least upper bound* for a set $S \subset \mathbb{R}$ if M is an upper bound for S and M is less than or equal to any other upper bound for S. The least upper bound for S is denoted by $\mathrm{l.u.b.}S$

The statement of our next theorem gives us the rationale (no pun intended) for developing the real numbers through Dedekind cuts.

Theorem

The set $\mathbb{R}$ of real numbers, constructed through Dedekind cuts, satisfies the least upper bound property; namely, if S is a nonempty subset of $\mathbb{R}$ that is bounded above, then there exists a least upper bound for S.

Having defined the elements of $\mathbb{R}$ through Dedekind cuts, we will now define the natural operations operations of addition, subtraction, multiplication, and division.

Definition

Let $x = A|B$ and $y = C|D$ be cuts. The sum of x and y is the cut $x + y = E|F$ where

$$E = \{r : r = a + c \text{ for some } a \in A \text{ and } c \in C\}$$

and $F = E'$ (where the complement is within $\mathbb{Q}$).

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- Is $E|F$, so defined, a cut?
- Does the cut addition agree with rational addition when x and y are rational cuts?

Definition

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Show that $0^* + x = x$ *for each cut* x

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Remark

The concepts of *positive*, *negative*, *nonnegative*, and *nonpositive* are defined in the obvious way with reference to 0^*.

Definition

Given the cut $x = A|B$, the *additive inverse* of x is the cut $-x = C|D$ such that

$$C = \{r : r = -b \text{ for some } b \in B, \text{ not the smallest element of } B\}$$

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Definition

Let x and y be a pair of cuts. We define the *subtraction* of cuts as follows:

$$x - y = x + (-y)$$

Definition

If $x = A|B$ and $y = C|D$ are nonnegative cuts, then $x \cdot y = xy = E|F$ where

$$E = \{r : r \leq 0 \text{ or } r = ac \text{ for some } a \in A \text{ and } b \in B \\ \text{where } a > 0 \text{ and } b > 0\}$$

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- If $x < 0$ and $y < 0$, then $xy = (-x)(-y)$.

Handwaving Alert!

The operation of cut addition is well defined, natural, commutative, associative, and has inverses with respect to the neutral element 0^*. The operation of cut multiplication is well defined, natural, commutative, associative, distributive over cut addition, and has inverses of nonzero elements with respect to the neutral element 1^*.

Field Axioms

The real numbers $\mathbb{R}$ satisfy the following axioms:

Property	Addition	Multiplication
Commutativity	$a + b = b + a$	$ab = ba$
Associativity	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Distributivity	$a(b + c) = ab + ac$	$(a + b)c = ac + bc$
Identity	$a + 0^* = a$	$a1^* = 1^*a$
Inverses	$a + (-a) = 0^*$	$aa^{-1} = 1^*$

Definition

The *magnitude* or *absolute value* of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

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Theorem (Triangle Inequality)

For all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

The completeness of $\mathbb{R}$

What happens if we carry out the construction of cuts on $\mathbb{R}$ (that is with $\mathbb{R}$ now in place of $\mathbb{Q}$)? Do we enlarge the set of real numbers through cuts? The answer is, "No". By virtue of the least upper bound property, every cut of real numbers occurs at a real number and can thus be identified with that real number. We say that the real numbers are *complete* in this sense.

Definition

Let S be a nonempty subset of $\mathbb{R}$.

- Let
$$\sup S = \begin{cases} \text{l.u.b.} S & \text{if } S \text{ is bounded above;} \\ +\infty & \text{if } S \text{ is not bounded above.} \end{cases}$$
We call $\sup S$ the *supremum* of S.
- Let
$$\inf S = \begin{cases} \text{g.l.b.} S & \text{if } S \text{ is bounded above;} \\ -\infty & \text{if } S \text{ is not bounded above.} \end{cases}$$
We call $\inf S$ the *infimum* of S.

Definition

A *sequence* of real numbers of real numbers is an ordered list of real numbers

$$a_1, a_2, a_3, \ldots$$

indexed by the natural numbers. More precisely, a sequence is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$.

Remark

Sometimes we will list the first few elements of the sequence with the idea that you will detect the underlying pattern. It is more precise, however, to give an explicit formula. For example, the sequences

$$1, 1/3, 1/5, 1/7, \ldots \qquad \text{and} \qquad (1/(2n-1),\ n \geq 1)$$

are identical, but the second representation is better.

Our next definition captures the notion that the values of a sequence may tend to approach a fixed number.

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Definition

The sequence (a_n) *converges* to the limit b as $n \to \infty$ provided that for each $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$

$$|a_n - b| < \varepsilon$$

When (a_n) converges to b, we will write either

$$\lim_{n\to\infty} a_n = b \quad \text{or} \quad a_n \to b.$$

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- Given any natural number M, a sequence (a_n) can be separated into a *head* and a *tail*:

$$\underbrace{a_1, a_2, a_3, \cdots, a_{M-1},}_{head} \underbrace{a_M, a_{M+1}, a_{M+2}, \cdots}_{tail}$$

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- Thus (a_n) converges to b if for any $\varepsilon > 0$, then ultimately a tail of the sequence is contained in the interval $(b - \varepsilon, b + \varepsilon)$.

Remark

The previous definition of convergence is unusable if the precise value of b is not known. Our next goal is to develop another characterization of convergence that will be helpful in this regard.

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A sequence (a_n) is said to satisfy *Cauchy's condition* provided that for any $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that

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Remark

If a sequence converges, then it satisfies Cauchy's condition.

Theorem

$\mathbb{R}$ is complete with respect to Cauchy sequences in the sense that if (a_n) is a sequence of real numbers obeying Cauchy's condition then it converges to a limit in $\mathbb{R}$.

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Theorem

A sequence (a_n) in $\mathbb{R}$ converges if and only if (a_n) satisfies Cauchy's condition.

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Example

The real number $\sqrt{2}$ is irrational.

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Theorem

Every interval of real numbers (a, b)*, no matter how small, contains both rational and irrational numbers.*

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Theorem

If x and y are real numbers and for each $\varepsilon > 0$, $|x - y| \leq \varepsilon$, then $x = y$.