Outline

1. Elementary number systems
2. An introduction to cuts
3. An ordering of cuts
4. Upper bounds and least upper bounds
5. Cut arithmetic
6. Magnitude
7. The completeness of the real numbers
8. Supremum and infimum
9. Cauchy sequences
10. Further description of \( \mathbb{R} \)
Elementary number systems

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Theorem

No number in \( \mathbb{Q} \) has a square equal to 2. In other words, \( \sqrt{2} \notin \mathbb{Q} \).
R. Dedekind (1831 - 1916)
**Definition**

A *cut* in \( \mathbb{Q} \) is a pair of subsets \( A, B \) of \( \mathbb{Q} \) such that

\[
A \cup B = \mathbb{Q}, \quad A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap B = \emptyset.
\]

If \( a \in A \) and \( b \in B \), then \( a < b \).

\( A \) contains no largest element.

In other words... The sets \( A \) and \( B \) must partition \( \mathbb{Q} \). \( A \) is to the "left" of \( B \).

**Notation**

We will write \( A | B \) for the cut with left-hand part \( A \) and right-hand part \( B \).
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Is \( A \upharpoonright B = \{ r : r < 3 \} \cup \{ r : r > 3 \} \) a cut in \( \mathbb{Q} \)?
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An introduction to cuts

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Is $A \cup B = \{ r : r < 1 \} \cup \{ r : r \geq 1 \}$ a cut in $\mathbb{Q}$?
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Is \( A \setminus B = \{ r : r < 3 \} \setminus \{ r : r > 3 \} \) a cut in \( \mathbb{Q} \)?

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Is \( A \setminus B = \{ r : r < 1 \} \setminus \{ r : r \geq 1 \} \) a cut in \( \mathbb{Q} \)?

**Definition**

We call the cut \( A \setminus B \) a *rational cut* if there exists a rational number \( c \) such that

\[
A \setminus B = \{ r : r < c \} \setminus \{ r : r \geq c \}
\]

We will write \( c^* \) to denote this rational cut.
Problem

Is \( A|B = \{ r : r \leq 0 \text{ or } r^2 < 2 \} \cup \{ r : r > 0 \text{ and } r^2 \geq 2 \} \) a cut in \( \mathbb{Q} \)?
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Problem

\text{Is this cut a rational cut?}
Definition

A real number is a cut in \( \mathbb{Q} \). We will let \( \mathbb{R} \) denote the class of cuts in \( \mathbb{Q} \).
Definition

Let \( x = A|B \) and \( y = C|D \) be a pair of cuts.

The cut \( x \) is equal to the cut \( y \) if \( A = C \); we write \( x = y \).

The cut \( x \) is less than or equal to the cut \( y \) if \( A \subset C \); we write \( x \leq y \).

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Is every pair of cuts $x$ and $y$ comparable? That is, must one of the three statements be true: $x = y$, $x < y$, or $y < x$?
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*Is every pair of cuts \( x \) and \( y \) comparable? That is, must one of the three statements be true:*

\[ x = y, \quad x < y \quad \text{or} \quad y < x? \]
Definition

A cut $M \in \mathbb{R}$ is an *upper bound* for a set $S \subset \mathbb{R}$ provided that $s \leq M$ for each $s \in S$. Equivalently we say that $S$ is *bounded above* by $M$. 
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Definition
A cut $M$ is a least upper bound for a set $S \subset \mathbb{R}$ if $M$ is an upper bound for $S$ and $M$ is less than or equal to any other upper bound for $S$. The least upper bound for $S$ is denoted by $\text{l.u.b.} S$. 
The statement of our next theorem gives us the rationale (no pun intended) for developing the real numbers through Dedekind cuts.

**Theorem**

The set \( \mathbb{R} \) of real numbers, constructed through Dedekind cuts, satisfies the least upper bound property; namely, if \( S \) is a nonempty subset of \( \mathbb{R} \) that is bounded above, then there exists a least upper bound for \( S \).
Having defined the elements of $\mathbb{R}$ through Dedekind cuts, we will now define the natural operations of addition, subtraction, multiplication, and division.

**Definition**

Let $x = A|B$ and $y = C|D$ be cuts. The sum of $x$ and $y$ is the cut $x + y = E|F$ where

$$E = \{ r : r = a + c \text{ for some } a \in A \text{ and } c \in C \}$$

and $F = E'$ (where the complement is within $\mathbb{Q}$).
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Definition

The zero cut is the rational cut $0^*$. 

Problem

Show that $0^* + x = x$ for each cut $x$.

Remark

The concepts of positive, negative, nonnegative, and nonpositive are defined in the obvious way with reference to $0^*$. 

Tom Lewis ()
§1.2–Cuts Fall Term 2006 14 / 28
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Given the cut $x = A \mid B$, the *additive inverse* of $x$ is the cut $-x = C \mid D$ such that

$$C = \{ r : r = -b \text{ for some } b \in B, \text{ not the smallest element of } B \}$$

Let $D = C'$

Problem

Show that $(-x) + x = 0^*$. 

Definition

Let $x$ and $y$ be a pair of cuts. We define the *subtraction* of cuts as follows:

$$x - y = x + (-y)$$
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Definition

If \( x = A \setminus B \) and \( y = C \setminus D \) are nonnegative cuts, then \( x \cdot y = xy = E \setminus F \)
where

\[
E = \{ r : r \leq 0 \text{ or } r = ac \text{ for some } a \in A \text{ and } b \in B \}
\]
where \( a > 0 \) and \( b > 0 \)

and \( F = E' \), the rest of \( \mathbb{Q} \).
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Definition

If \( x \) and \( y \) are not both nonnegative, then the definition of \( xy \) can be extended in obvious ways:

\[
\begin{align*}
\text{If } x \geq 0 \text{ and } y < 0, \quad xy &= -x(-y) \\
\text{If } x < 0 \text{ and } y \geq 0, \quad xy &= -(x(-y)) \\
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**Definition**

If \( x \) and \( y \) are not both nonnegative, then the definition of \( xy \) can be extended in obvious ways:

- If \( x \geq 0 \) and \( y < 0 \), then \( xy = - (x(-y)) \)
- If \( x < 0 \) and \( y \geq 0 \), then \( xy = -((-x)y) \)
- If \( x < 0 \) and \( y < 0 \), then \( xy = (-x)(-y) \).
The operation of cut addition is well defined, natural, commutative, associative, and has inverses with respect to the neutral element $0^*$. The operation of cut multiplication is well defined, natural, commutative, associative, distributive over cut addition, and has inverses of nonzero elements with respect to the neutral element $1^*$. 

Handwaving Alert!
## Field Axioms

The real numbers $\mathbb{R}$ satisfy the following axioms:

<table>
<thead>
<tr>
<th>Property</th>
<th>Addition</th>
<th>Multiplication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutativity</td>
<td>$a + b = b + a$</td>
<td>$ab = ba$</td>
</tr>
<tr>
<td>Associativity</td>
<td>$(a + b) + c = a + (b + c)$</td>
<td>$(ab)c = a(bc)$</td>
</tr>
<tr>
<td>Distributivity</td>
<td>$a(b + c) = ab + ac$</td>
<td>$(a + b)c = ac + bc$</td>
</tr>
<tr>
<td>Identity</td>
<td>$a + 0^* = a$</td>
<td>$a1^* = 1^*a$</td>
</tr>
<tr>
<td>Inverses</td>
<td>$a + (−a) = 0^*$</td>
<td>$aa^{-1} = 1^*$</td>
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Definition

The *magnitude* or *absolute value* of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0.
\end{cases}$$
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\]

Theorem (Triangle Inequality)

*For all \( x, y \in \mathbb{R} \), \( |x + y| \leq |x| + |y| \).*
The completeness of $\mathbb{R}$

What happens if we carry out the construction of cuts on $\mathbb{R}$ (that is with $\mathbb{R}$ now in place of $\mathbb{Q}$)? Do we enlarge the set of real numbers through cuts? The answer is, “No”. By virtue of the least upper bound property, every cut of real numbers occurs at a real number and can thus be identified with that real number. We say that the real numbers are complete in this sense.
Definition

Let $S$ be a nonempty subset of $\mathbb{R}$.

- Let

$$\sup S = \begin{cases} 
\text{l.u.b.} S & \text{if } S \text{ is bounded above;} \\
+\infty & \text{if } S \text{ is not bounded above.}
\end{cases}$$

We call $\sup S$ the *supremum* of $S$.

- Let

$$\inf S = \begin{cases} 
\text{g.l.b.} S & \text{if } S \text{ is bounded above;} \\
-\infty & \text{if } S \text{ is not bounded above.}
\end{cases}$$

We call $\inf S$ the *infimum* of $S$.  

Definition
A sequence of real numbers of real numbers is an ordered list of real numbers

\[ a_1, a_2, a_3, \ldots \]

indexed by the natural numbers. More precisely, a sequence is a function from the natural numbers \( \mathbb{N} \) to the real numbers \( \mathbb{R} \).

Remark
Sometimes we will list the first few elements of the sequence with the idea that you will detect the underlying pattern. It is more precise, however, to give an explicit formula. For example, the sequences

\[ 1, 1/3, 1/5, 1/7, \ldots \]

\[ (1/(2n - 1), \ n \geq 1) \]

are identical, but the second representation is better.
Our next definition captures the notion that the values of a sequence may tend to approach a fixed number.
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**Definition**

The sequence \((a_n)\) **converges** to the limit \(b\) as \(n \to \infty\) provided that for each \(\varepsilon > 0\) there exists a natural number \(N \in \mathbb{N}\) such that for all natural numbers \(n \geq N\)

\[ |a_n - b| < \varepsilon \]

When \((a_n)\) converges to \(b\), we will write either

\[
\lim_{n \to \infty} a_n = b \quad \text{or} \quad a_n \to b.
\]
An alternative characterization

Observe that $|a_n - b| < \varepsilon$ if and only if $a_n \in (b - \varepsilon, b + \varepsilon)$.

Given any natural number $M$, a sequence $(a_n)$ can be separated into a head and a tail:

$$a_1, a_2, a_3, \ldots, a_{M-1}, a_M, a_{M+1}, a_{M+2}, \ldots$$

Thus $(a_n)$ converges to $b$ if for any $\varepsilon > 0$, then ultimately a tail of the sequence is contained in the interval $(b - \varepsilon, b + \varepsilon)$. 
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\[
\underbrace{a_1, a_2, a_3, \ldots, a_{M-1}}_{\text{head}} \quad \underbrace{a_M, a_{M+1}, a_{M+2}, \ldots}_{\text{tail}}
\]

- Thus \((a_n)\) converges to \( b \) if for any \( \varepsilon > 0 \), then ultimately a tail of the sequence is contained in the interval \((b - \varepsilon, b + \varepsilon)\).
Remark

The previous definition of convergence is unusable if the precise value of $b$ is not known. Our next goal is to develop another characterization of convergence that will be helpful in this regard.
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Definition
A sequence $(a_n)$ is said to satisfy Cauchy’s condition provided that for any $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that

$$\text{if } m, n \geq N, \text{ then } |a_n - a_m| < \varepsilon$$
Remark

The previous definition of convergence is unusable if the precise value of $b$ is not known. Our next goal is to develop another characterization of convergence that will be helpful in this regard.

Definition

A sequence $(a_n)$ is said to satisfy Cauchy’s condition provided that for any $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that

$$\text{if } m, n \geq N, \text{ then } |a_n - a_m| < \varepsilon$$

Remark

If a sequence converges, then it satisfies Cauchy’s condition.
Theorem

\( \mathbb{R} \) is complete with respect to Cauchy sequences in the sense that if \((a_n)\) is a sequence of real numbers obeying Cauchy’s condition then it converges to a limit in \( \mathbb{R} \).
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Theorem

A sequence \((a_n)\) in \( \mathbb{R} \) converges if and only if \((a_n)\) satisfies Cauchy's condition.
Definition
An *irrational* number is a non-rational real number.
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Example
The real number $\sqrt{2}$ is irrational.
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Example

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Theorem

Every interval of real numbers $(a, b)$, no matter how small, contains both rational and irrational numbers.
Our next result is often called the Archimedean property of the real numbers.
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**Theorem**

*For each* $x \in \mathbb{R}$, *there is an integer* $n \geq x$.
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**Theorem**

*If* $x$ *and* $y$ *are real numbers and for each* $\varepsilon > 0$, $|x - y| \leq \varepsilon$, *then* $x = y$. 