6.4–Properties of Multivariate Distributions

Tom Lewis

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Outline

1. The joint distribution function

2. Independent variables

3. Sums of independent random variables
The joint distribution function

**Definition**

- Let $X_1, X_2, \cdots, X_n$ be $n$ random variables defined on a common probability space. Their *joint distribution function* $F$ is defined by
  
  $$F(x_1, x_2, \cdots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n)$$

- The *marginal distribution* of each $X_i$ is defined by
  
  $$F_{X_i}(x_i) = P(X_i \leq x_i), \quad i = 1, 2, \cdots, n.$$}

**Remark**

The marginal distribution function of $X_i$ can always be recovered from the joint distribution function

$$F(x_1, x_2, \cdots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n)$$

by sending all of the variables other than $x_i$ to $+\infty$. 

\[\text{}\]
### Definition

- A nonnegative function $f(u_1, u_2, \ldots, u_n)$ is said to be a density for the distribution function $F(x_1, x_2, \ldots, x_n)$ provided that

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(u_1, u_2, \ldots, u_n) \, du_1 du_2 \cdots du_n$$

- If the distribution function of the random variables $(X_1, \ldots, X_n)$ has density $f(u_1, \ldots, u_n)$ and if $A \subset \mathbb{R}^n$ is a “nice” set, then

$$P((X_1, \ldots, X_n) \in A) = \int_A f(u_1, \ldots, u_n) \, du_1 du_2 \cdots du_n.$$  

### Independent variables

- The random variables $(X_1, \ldots, X_n)$ are independent if and only if their joint distribution is the product of the marginal distribution functions.

- If the random variables $(X_1, \ldots, X_n)$ have a density $f(u_1, \ldots, u_n)$ then they are independent if and only if $f$ is the product of the marginal density functions, that is,

$$f(u_1, u_2, \ldots, u_n) = f_{X_1}(u_1) f_{X_2}(u_2) \cdots f_{X_n}(u_n).$$
Theorem

Let $X_1, \cdots, X_n$ be independent random variables such that $X_i$ has the normal density $N(\mu_i, \sigma_i^2)$, $1 \leq i \leq n$. Then $X_1 + \cdots + X_n$ has the normal density $N(\mu, \sigma^2)$ where

$\mu = \mu_1 + \cdots + \mu_n$ \text{ and } $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

Theorem

Let $X_1, \cdots, X_n$ be independent random variables such that $X_i$ has the gamma density $\Gamma(\alpha_i, \lambda)$, $1 \leq i \leq n$. Then $X_1 + \cdots + X_n$ has the gamma density $\Gamma(\alpha, \lambda)$ where

$\alpha = \alpha_1 + \cdots + \alpha_n$.

Problem

Let $X_1, \cdots, X_n$ be a sequence of independent $N(0, 1)$ random variables. What is the distribution of

$X_1^2 + \cdots + X_n^2$?

Solution

- Recall that each $X_i^2$ has a $\Gamma(1/2, 1/2)$ density.
- According to our theorem, the sum of the squares has a $\Gamma(n/2, 1/2)$ density.