§4.6 Chebyshev’s Inequality

Tom Lewis

Fall Term 2008
Outline

1. Review

2. Chebyshev’s inequality

3. The weak law of large numbers
**Theorem**

*If $X \geq Y$, then $E(X) \geq E(Y)$.***
Theorem

If \( X \geq Y \), then \( E(X) \geq E(Y) \).

Theorem (Markov’s inequality)

If \( X \) have a first moment if \( t > 0 \), then

\[
P(|X| \geq t) \leq \frac{E(|X|)}{t}.
\]
Proof.

Let \( Z = \begin{cases} t & \text{if } |X| \geq t \\ 0 & \text{if } |X| < t \end{cases} \).

Observe that \( |X| \geq Z \).

Thus \( E(|X|) \geq E(Z) = tP(|X| \geq t) \), as was to be shown.
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Observe that $|X| \geq Z$. Thus

$$E(|X|) \geq E(Z) = tP(|X| \geq t),$$

as was to be shown.
Theorem (Chebyshev’s Inequality)

If $X$ has a second moment, $\mu = E(X)$, and $t > 0$, then

$$P(|X - \mu| \geq t) \leq \frac{\text{var}(X)}{t^2}.$$
Theorem (Chebyshev’s Inequality)

If $X$ has a second moment, $\mu = E(X)$, and $t > 0$, then

$$P(|X - \mu| \geq t) \leq \frac{\text{var}(X)}{t^2}.$$ 

Proof.

We will use Markov’s inequality:

$$P(|X - \mu| \geq t) = P(|X - \mu|^2 \geq t^2) \leq \frac{E((X - \mu)^2)}{t^2} = \frac{\text{var}(X)}{t^2},$$
Problem

Toss 100 coins and let $H$ count the number of heads. Use Chebyshev’s inequality to estimate the probability that $40 \leq H \leq 60$. 

Solution

We have

$$P\left(\left|H - 50\right| > 10\right) \leq \frac{\text{var}(H)}{10^2} = \frac{100(1/2)(1/2)}{100} = \frac{1}{4}$$

Thus, $P(40 \leq H \leq 60) \geq \frac{3}{4}$.

This is a quick but very bad estimate.
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Theorem (The Weak Law of Large Numbers)

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with second moments and identical density functions. Let $\mu = \mathbb{E}(X_1)$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then for all $\varepsilon > 0$, $\Pr(\left| S_n - \mu \right| \geq \varepsilon) \to 0$ as $n \to \infty$. 
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**Theorem (The Weak Law of Large Numbers)**

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- Let $\mu = E(X_1)$.
- Let $S_n = X_1 + X_2 + \cdots + X_n$. 

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Theorem (The Weak Law of Large Numbers)

- Let $X_1, X_2, \ldots$ be a sequence of independent random variables with second moments and identical density functions.
- Let $\mu = E(X_1)$.
- Let $S_n = X_1 + X_2 + \cdots + X_n$.
- Then for all $\varepsilon > 0$,

$$P \left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) \to 0$$

as $n \to \infty$. 
Proof.

We will use Chebyshev’s inequality:

\[ P \left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) = P(\left| S_n - n\mu \right| \geq n\varepsilon) \]

\[ \leq \frac{\text{var}(S_n)}{n^2\varepsilon^2} \]

\[ = \frac{n\text{var}(X_1)}{n^2\varepsilon^2} \]

\[ = \frac{\text{var}(X_1)}{n\varepsilon^2}. \]

This last term converges to 0 as \( n \to \infty \), as was to be shown.