1. In each of the following problems, you will be given a mathematical definition. Express this definition using the quantifiers \( \forall \) and \( \exists \), negate the definition, and write the negation of the definition in standard English.

   (a) \( f \) is integrable on \([a, b]\) provided that for every \( \varepsilon > 0 \) there is a partition \( P \) of \([a, b]\) such that \( U(P, f) - L(P, f) < \varepsilon \).

   \[ f \text{ is integrable on } [a, b] \text{ provided that } \forall \varepsilon > 0, \exists \text{ partition } P \subseteq [a, b], \quad U(P, f) - L(P, f) < \varepsilon \]

   \[ f \text{ is not integrable on } [a, b] \text{ provided that } \exists \varepsilon > 0, \forall \text{ partition } P \subseteq [a, b], \quad U(P, f) - L(P, f) \geq \varepsilon \]

   \[ f \text{ is not integrable on } [a, b] \text{ provided that there exists an } \varepsilon > 0 \text{ such that for all partitions } P \subseteq [a, b], \quad U(P, f) - L(P, f) \geq \varepsilon \]

   (b) The sequence \( \{a_n\} \) converges to the \( L \) provided that for every \( \varepsilon > 0 \) there is a natural number \( N \) such that if \( n \geq N \), then \( |a_n - L| < \varepsilon \).

   \[ \text{The sequence } \{a_n\} \text{ converges to } L \text{ provided that } \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N \text{ implies } |a_n - L| < \varepsilon. \]

   \[ \text{The sequence } \{a_n\} \text{ does not converge to } L \text{ provided that } \exists \varepsilon > 0, \forall N \in \mathbb{N}, \ \exists n > N \text{ and } |a_n - L| \geq \varepsilon. \]

   \[ \text{The sequence } \{a_n\} \text{ does not converge to } L \text{ provided that there exists an } \varepsilon > 0 \text{ such that for every natural number } N \text{ we have } n \geq N \text{ and } |a_n - L| \geq \varepsilon. \]
2. Prove the following theorem: \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 0 \).

Let \( x \in \mathbb{Z} \). We must show that there exists \( y \in \mathbb{Z} \) such that \( x + y = 0 \).

Let \( y = -x \). Then \( x + y = x + (-x) = 0 \), as was to be shown.

3. Prove or disprove: \( A \cup B = A \cap B \) if and only if \( A = B \).

This is true.

If \( A = B \), then \( A \cup B = A \) and \( A \cap B = A \)

Hence \( A \cup B = A \cap B \).

Conversely, let us assume \( A \cup B = A \cap B \). We will show \( A = B \).

Let \( x \in A \). Then \( x \in A \cup B \). But \( A \cup B = A \cap B \); thus, \( x \in A \cap B \).

But \( A \cap B \subseteq B \); thus, \( x \in B \), and \( A \subseteq B \).

By reversing the roles of \( A \) and \( B \), we can see that \( B \subseteq A \) as well. Thus \( A = B \).

4. How many integers between 1 and 1000 inclusive are divisible by 5 or 13?

\[
A = \{ j \in \mathbb{Z} : 1 \leq j \leq 1000 \text{ and } 5 \mid j \}\n\]

\[
B = \{ k \in \mathbb{Z} : 1 \leq k \leq 1000 \text{ and } 13 \mid k \}\n\]

Then \( |A \cup B| = |A| + |B| - |A \cap B| \), by inclusion/exclusion principle.

But \( |A| = 200 \) and \( |B| = 76 \).

\( A \cap B = \{ r \in \mathbb{Z} : 1 \leq r \leq 1000 \text{ and } \gcd(r, 5) = \gcd(r, 13) = 1 \}\)

Hence \( A \cap B = 15 \).

Thus \( |A \cup B| = 200 + 76 - 15 = 261 \).
5. Let $O$ be the set of odd integers and let $A = \{ x \in \mathbb{Z} : \exists k \in \mathbb{Z}, x = 4k+3 \}$. Show that $A \subseteq O$.

Let $x \in A$. Then $x = 4k+3$ for some $k \in \mathbb{Z}$. But

$$x = 4k+3 = 4k+2+1 = 2(2k+1)+1.$$ 

Since $2k+1 \in \mathbb{Z}$, $x$ is an odd number and $x \in O$. This shows $A \subseteq O$.

6. Prove the following Law of DeMorgan: Let $A$, $B$, and $C$ be sets. Then

$$A - (B \cup C) = (A - B) \cap (A - C).$$

$$A - (B \cup C) = \{ x : (x \in A) \land \neg (x \in B \lor x \in C) \}$$

$$= \{ x : (x \in A) \land (x \notin B) \land (x \notin C) \}$$

$$= \{ x : (x \in A) \land (x \notin B) \land (x \notin C) \}$$

$$= \{ x : (x \in A) \land (x \notin B) \} \cap \{ x : (x \in A) \land (x \notin C) \}$$

$$= (A - B) \cap (A - C).$$