§22 Recurrence Relations

Tom Lewis

Fall Term 2010

Outline

1. Introduction to recurrence relations
2. The structure of first-order linear recurrence relations
3. Second-order linear recurrence relations
4. The structure of solutions of second-order recurrence relations
5. Examples

Motivation

Sometimes it easier to describe a sequence $a_0, a_1, a_2, \cdots$ in terms of itself (recursively) rather than in absolute terms.

Problem

At the beginning of a month, Jane invests \$1000.00 dollars into an account. Thereafter Jane pays an additional \$100.00 into this account at the end of each month. The account pays 5% interest at the end of each month. Let P_n denote the amount of money in the account at the end of the nth month.

- *Find P_1, P_2, and P_3.*
- *Find a recurrence equation relating P_n to P_{n-1}.*

First-order recurrence relations

- Let s and t be real numbers. The recursive relation

$$a_n = sa_{n-1} + t \tag{1}$$

  is called a first-order linear recurrence relation.
- If we specify $a_0 = \alpha$, then we call α an initial condition.

Theorem (Uniqueness of solutions)

If an initial condition is specified for the first-order linear recurrence relation (1), then this equation has a unique solution.

Lemma

The unique solution of the first-order recurrence relation

$$a_n = sa_{n-1} + t, \quad a_0 = \alpha$$

is: $a_0 = \alpha$ *and*

$$a_n = s^n\alpha + t(1 + s + s^2 + \cdots + s^{n-1}) \quad \text{for } n \geq 1.$$

Theorem

The first-order recurrence relation

$$a_n = sa_{n-1} + t, \quad a_0 = \alpha$$

has a unique solution; namely, $a_0 = \alpha$ *and, for each* $n \geq 1$,

$$a_n = \begin{cases} s^n\alpha + t\dfrac{s^n - 1}{s - 1} & \text{if } s \neq 1; \\ \alpha + nt & \text{if } s = 1 \end{cases}$$

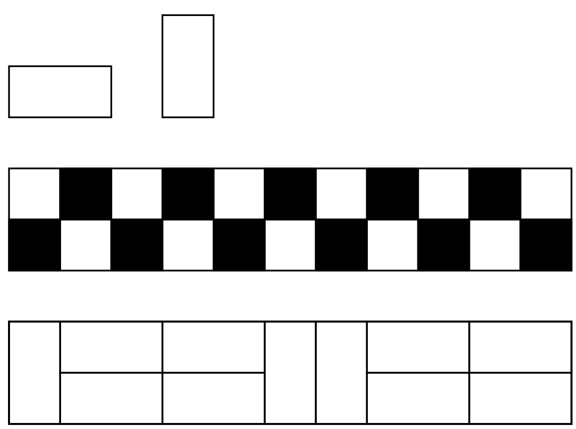
Problem

At the beginning of a month, Jane invests \$1000.00 dollars into an account. Thereafter Jane pays an additional \$100.00 into this account at the end of each month. The account pays 5% interest at the end of each month. Let P_n denote the amount of money in the account at the end of the nth month. Solve the first-order linear recurrence equation and find a closed-form formula for P_n.

Problem

In how many ways can a set of dominoes tile a $2 \times n$ checkerboard?

Figure: One of the ways to tile a 2×11 checkerboard with a set of dominoes



Second-order linear recurrence relations

- Let s_1 and s_2 be real numbers. The recursive relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2} \tag{2}$$

  is called a second-order linear homogeneous recurrence relation.
- If we specify $a_0 = \alpha_0$ and $a_1 = \alpha_1$, then we call α_0 and α_1 the initial conditions.

Theorem (Uniqueness of solutions)

If initial conditions are specified for the second-order linear recurrence relation (2), then this equation has a unique solution.

Problem

Show that the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

has solutions

- $a_n = 2^n$
- $a_n = 3^n$
- $a_n = C_1 2^n + C_2 3^n$, *where* C_1 *and* C_2 *are arbitrary constants.*

Problem

Find the solution of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad a_0 = 1 \text{ and } a_1 = 6$$

Problem

If $a_n = r^n$ is a solution of the recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2},$$

then what condition must r satisfy?

The characteristic polynomial

The characteristic polynomial of the second-order recurrence relation

$$a_n = s_1 a_{n-1} + s_2 a_{n-2}$$

is given by $p(x) = x^2 - s_1 x - s_2$.

Theorem

If r is a root of the characteristic polynomial $p(x)$ and C is any real number, then $a_n = Cr^n$ solves the second-order recurrence relation (2).

Problem

Recall the recurrence relation related to the tiling of the $2 \times n$ checkerboard by dominoes:

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, \quad a_2 = 2$$

- *Find the characteristic polynomial and determine its roots.*
- *Solve the recurrence relation with its initial conditions.*

Rooting interest

The solution of a second-order linear recurrence relation depends upon the structure of the roots of the characteristic polynomial.

The trichotomy

The roots of the characteristic polynomial can fall into one and only one the following cases:

Distinct real roots There can be two distinct real roots: r_1, r_2.

Complex roots There can be two distinct complex roots: z_1, z_2.

Repeated real root There can be a single repeated root: r.

Theorem (Distinct real roots)

Let the second-order linear recurrence relation (2) with initial conditions $a_1 = \alpha_0$ and $a_1 = \alpha_1$ be given. If the characteristic polynomial has two distinct real roots r_1 and r_2, then the solution of the recurrence relation is given by

$$a_n = C_1 r_1^n + C_2 r_2^n, \quad \text{for } n \geq 0,$$

where C_1 and C_2 are the solutions of the equations:

$$\begin{array}{ccccc} C_1 & + & C_2 & = & \alpha_0 \\ C_1 r_1 & + & C_2 r_2 & = & \alpha_1 \end{array}$$

Theorem (Complex roots)

Let the second-order linear recurrence relation (2) with initial conditions $a_1 = \alpha_0$ and $a_1 = \alpha_1$ be given. If the characteristic polynomial has the complex roots z_1 and z_2, then the solution of the recurrence relation is given by

$$a_n = C_1 z_1^n + C_2 z_2^n, \quad \text{for } n \geq 0,$$

where C_1 and C_2 are the solutions of the equations:

$$\begin{array}{rcrcl} C_1 & + & C_2 & = & \alpha_0 \\ C_1 z_1 & + & C_2 z_2 & = & \alpha_1 \end{array}$$

Theorem (Repeated real root)

Let the second-order linear recurrence relation (2) with initial conditions $a_1 = \alpha_0$ and $a_1 = \alpha_1$ be given. We suppose that the characteristic polynomial has a repeated root r. There are two cases:

1. *If $r \neq 0$, then the solution of the recurrence relation is given by*

   $$a_n = C_1 r^n + C_2 n r^n, \quad \text{for } n \geq 0,$$

   where C_1 and C_2 are the solutions of the equations:

   $$\begin{array}{rcrcl} C_1 & & & = & \alpha_0 \\ C_1 r & + & C_2 r & = & \alpha_1 \end{array}$$

2. *If $r = 0$, then $a_n = 0$ for all $n \geq 2$.*

Problem

Solve the following recurrence relations:

1. $a_n = 6a_{n-1} - 9a_{n-2},\ a_0 = 2,\ a_1 = 21$
2. $a_n = 4a_{n-1} - 5a_{n-1},\ a_0 = 2,\ a_1 = 6$