§21 Induction

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Outline

1. The method of induction

2. Strong mathematical induction
Pessimists and Optimists

In the solving of problems, the method of smallest counterexample is a “glass-half-empty” approach; mathematical induction is a “glass-half-full” approach.
The method of induction

Tipping propositions with mathematical induction

Suppose that we want to show that each of the propositions $P_0, P_1, P_2, \ldots$ is true.

First show that $P_0$ is true. This is the customary base case or basis step.

Here is the tricky part. If we can show that for each $k \geq 0$, $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true.

The Induction Hypothesis

then, in fact, we will shown that each $P_k$, $k \in \mathbb{N}$, is true.
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$$ P_k \text{ is true} \implies P_{k+1} \text{ is true.} $$

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Problem

*Use mathematical induction to prove the following theorem.*
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Use mathematical induction to prove the following theorem.

Theorem

Let \( n \) be a natural number. Then

\[
0 + 1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.
\]
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Theorem

Let $n$ be a natural number. Then

$$1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$
Problem

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Theorem

Let $n \geq 1$ be an integer. Then $n < 2^n$. 
Problem

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Theorem
Let $n$ be a natural number. Then $4^n - 1$ is divisible by 3.
The method of induction

Problem

Let \( \{a_n\} \) be a sequence of numbers defined recursively as: \( a_0 = 10 \) and, for \( n \geq 1 \),

\[
a_n = \frac{1}{2}a_{n-1} + 10.
\]

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Use mathematical induction to prove the following theorem.

Theorem

Let \( n \) be a natural number. Then \( a_n \leq 20 \).
Tiling with L-triominos
Theorem

Let $n \geq 1$ be an integer. A set of L-triominos can tile all but one cell of a $2^n \times 2^n$ chessboard.
Suppose that we want to show that each of the propositions $P_0$, $P_1$, $P_2$, … is true.

First show that $P_0$ is true. This is the customary base case or basis step.

Here is the tricky part. If we can show that for each $k \geq 0$, $P_0$, $P_1$, …, $P_k$ are true \[ \Rightarrow \] The Strong Induction Hypothesis

then, in fact, we will shown that each $P_k$, $k \in \mathbb{N}$, is true.
Tipping propositions with mathematical induction

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Tipping propositions with mathematical induction

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$$P_0, P_1, \ldots, P_k \text{ are true} \quad \Rightarrow \quad P_{k+1} \text{ is true.}$$

The Strong Induction Hypothesis

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Strong mathematical induction

Problem

Define a sequence \( \{g_n\} \) as follows: \( g_1 = 5 \), \( g_2 = 13 \), and

\[
g_n = 5g_{n-1} - 6g_{n-2} \quad \forall n \geq 3.
\]

Show that \( g_n = 2^n + 3^n \) for all \( n \geq 1 \).