Outline

1. The method of smallest counterexample

2. The well-ordering principle
A joke with a purpose

Theorem

All of the natural numbers are interesting.

A step-stone

We proved this theorem in the previous section.

Theorem

An integer cannot be both even and odd.
Another stone

Theorem

Every natural number is either even or odd.

Our destination

Theorem

Every integer is either even or odd.
The method of the smallest counterexample

- You wish to prove a theorem of the form: \( \forall i \geq 0, P_i \) is true, where each \( P_i \) is a statement.
- We proceed by contradiction. The negation of the theorem’s statement is: \( \exists x \geq 0 \) such that \( P_x \) is false.
- Consider the smallest \( i \geq 0 \) such that \( P_i \) is false. Call it \( x \). We will call \( P_x \) the smallest counterexample.
- Show that \( P_0 \) is true. This establishes the base case or basis step. We may conclude that \( x \geq 1 \). This is important!
- We now know that “\( P_x \) is false” and “\( P_{x-1} \) is true”. Show that this leads to a contradiction.
- The last step is often the difficult step. It may require Herculean ingenuity.

Theorem

Let \( n \) be a positive integer. The sum of the first \( n \) odd natural numbers is \( n^2 \). In other words

\[
1 + 3 + 5 + \cdots (2k - 1) = k^2 \quad \text{for all } k \geq 1.
\]
The well-ordering principle
Every non-empty set of natural numbers contains a least element.

Some things to note
You cannot alter any of the elements of this theorem without endangering the result. Why do neither of the following sets have a least element?
- Let $A = \{z \in \mathbb{N} : z \text{ is even and odd}\}$.
- Let $B = \{r \in \mathbb{Q} : r > 0\}$.

Proofs using the Well-Ordering Principle
- Let $P_n$ be a statement about the $n$th natural number.
- You want to prove the theorem: $\forall n \in \mathbb{N}, P_n$ is true.
- We suppose that this is not true, for the sake of a contradiction.
- Let $X = \{n \in \mathbb{N} : P_n \text{ is false}\}$. Our assumption is that $X \neq \emptyset$.
- Show that $0 \notin X$. This is the basis step.
- By WOP, let $x$ be the least element of $X$.
- Show that “$P_x$ is false” and “$P_{x-1}$ is true” leads to a contradiction.
Problem

Use the Well-Ordering Principle to show that

\[ \forall n \geq 5, \quad 2^n > n^2. \]
Definition (The Fibonacci numbers)

The Fibonacci numbers are the list of integers $F_0, F_1, F_2, \ldots$ defined recursively according to the rule: $F_0 = 1$, $F_1 = 1$ and, for each $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$ 

Problem

Find $F_7$.

A stronger form of proof by WOP

So far we have been using the following fact: if $x$ is the least element of $X$, then

$$P_x \text{ is false and } P_{x-1} \text{ is true.}$$

In fact, we can assert something much stronger. If $x$ is the least element of $X$, then

$$P_x \text{ is false and } P_0, P_1, P_2, \ldots, P_{x-1} \text{ are true.}$$

Theorem

For $n \in \mathbb{N}$, $F_n \leq (1.7)^n$. 