# §20 Smallest Counterexample

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# Outline

## A joke with a purpose

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### Theorem

*All of the natural numbers are interesting.*

## A step-stone

We proved this theorem in the previous section.

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*An integer cannot be both even and odd.*

## Another stone

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### Theorem

*Every natural number is either even or odd.*

### Our destination

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- The last step is often the difficult step. It may require Herculean ingenuity.

### Theorem

*Let n be a positive integer. The sum of the first n odd natural numbers is $n^2$. In other words*

$$1 + 3 + 5 + \cdots (2k - 1) = k^2 \quad \textit{for all } k \geq 1.$$

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- Let $A = \{z \in \mathbb{N} : z \text{ is even and odd}\}$.
- Let $B = \{r \in Q : r > 0\}$.

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- Show that $0 \notin X$. This is the basis step.
- By WOP, let $x$ be the least element of $X$.
- Show that "$P_x$ is false" and "$P_{x-1}$ is true" leads to a contradiction.

## Problem

*Use the Well-Ordering Principle to show that*

$$\forall n \geq \mathbb{N}, \quad 1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

## Problem

*Use the Well-Ordering Principle to show that*

$$\forall n \geq 5, \quad 2^n > n^2.$$

**Definition (The Fibonacci numbers)**

The Fibonacci numbers are the list of integers $F_0, F_1, F_2, \ldots$ defined recursively according to the rule: $F_0 = 1$, $F_1 = 1$ and, for each $n \geq 2$,

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## Problem

*Find* $F_7$.

## A stronger form of proof by WOP

So far we have been using the following fact: if $x$ is the least element of $X$, then

$$P_x \text{ is false and } P_{x-1} \text{ is true.}$$

In fact, we can assert something much stronger. If $x$ is the least element of $X$, then

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## Theorem

*For $n \in \mathbb{N}$, $F_n \leq (1.7)^n$.*