§19 Contradiction

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Outline

1 Proof by contrapositive

2 Reductio ad absurdum
Proof by contrapositive

The contrapositive

The contrapositive of the statement “If A, then B” is the statement “If not B, then not A”.

Theorem

$p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.

Proof by contrapositive

Proving $p \rightarrow q$ is equivalent to proving its contrapositive, $\neg q \rightarrow \neg p$. This is especially helpful when a theorem has been posed in a “negative” form.

Theorem

Let $R$ be an equivalence relation on a set $A$ and let $a, b \in A$. Show that if $b \notin [a]$, then $a \notin [b]$.

Theorem

Let $f(x) = mx + b$ be a function from $\mathbb{R}$ to $\mathbb{R}$. If $x \neq y$, then $f(x) \neq f(y)$. 
A linguistic note

Reduction ad absurdum is Latin for “reduce to the absurd.” In English we often call this line of attack proof by contradiction.

Reductio ad absurdum in 3 easy steps

You want to prove statement $s$ is true.

1. Either $s$ is true or $s$ is false.
2. Suppose that you can create a true chain of implications

   $$\neg s \rightarrow \cdots \rightarrow \text{FALSE},$$

   then $\neg s$ must be FALSE as well. This is the reductio ad absurdum part: denying $s$ leads to a FALSE conclusion, something absurd.
3. But if $\neg s$ is false, then $s$ must be true.

G.H. Hardy

Reductio ad absurdum, which Euclid loved so much, is one of a mathematician’s finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.
Tell the reader

- Starting a proof by contradiction without warning can have a devastating psychological affect on the reader.
- Alert the reader of your intentions! Start your proof with something like “For the sake of a contradiction, . . .,” or “We will proceed by contradiction . . .” or ”In order to arrive at a contradiction . . .” or “Assume, to the contrary, that . . .”

Problem

If $n$ is an integer, then $n$ cannot be both even and odd.

Problem

Let $R$ be an equivalence relation on a set $A$. If $(a, b) \notin R$, then $[a] \cap [b] = \emptyset$.

Problem

$\sqrt{2}$ is not a rational number.

Problem

The length of the hypotenuse of a right triangle is strictly than the sum of the lengths of the other two sides.

Problem

There are no positive, integer solutions of the equation $x^2 - y^2 = 10$. (We will assume basic facts about prime numbers here; we will shore up this foundation later).