§18 Inclusion-Exclusion

Tom Lewis

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Outline

1. Counting unions

2. The general form of inclusion-exclusion

3. Counting derrangements
Recall

For sets $A$ and $B$,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can this principle be extended?

Theorem (Inclusion-Exclusion)

For three sets $A$, $B$, and $C$,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$
The general form of inclusion-exclusion

Theorem (Inclusion-Exclusion)

Let \( A_1, A_2, \ldots, A_n \) be sets. Then

\[
|A_1 \cup A_2 \cup \cdots \cup A_n| = l_1 - l_2 + l_3 - l_4 + \cdots (-1)^{n+1} l_n
\]

where

\[
\begin{align*}
l_1 & = \text{the sum of the sizes of the sets taken 1 at a time} \\
l_2 & = \text{the sum of the sizes of all intersections taken 2 at a time} \\
l_3 & = \text{the sum of the sizes of all intersections taken 3 at a time} \\
& \vdots \\
l_{n-1} & = \text{the sum of the sizes of all intersections taken } n-1 \text{ at a time} \\
l_n & = \text{the size of the intersection of all } n \text{ sets}
\end{align*}
\]

Counting derrangements

Problem (The secretary’s problem)

A secretary has \( n \) addressed letters and \( n \) addressed envelopes. Each letter corresponds to exactly one envelope. In how many ways can the secretary place the letters in the envelopes so that each envelope contains exactly one letter, but no envelope contains its proper letter? Such an arrangement is called a derrangement.