§14 Equivalence Relations

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Outline

1. The definition
2. Congruence modulo $n$
3. Has-the-same-size-as
4. Equivalence classes
5. Some theorems
6. Partitioning by equivalence classes
Equivalence relations are ubiquitous. They are like the air we breathe; we hardly take notice.

### Example
- Equivalence of logical propositions.
- Congruence of triangles in geometry.
- Equivalence of systems of equations in linear algebra.
- Equivalence of measurable functions in analysis.
- Equivalence of time on the clock.

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**Definition (Equivalence relation)**

Let $R$ be a relation on a set $A$. We say $R$ is an equivalence relation provided it is reflexive, symmetric, and transitive. Thus:

- $aRa$ for all $a \in A$.
- $aRb$ implies $bRa$ for all $a, b \in A$.
- If $aRb$ and $bRc$, then $aRc$. 
Definition
Let $n$ be a positive integer. We say that the integers $x$ is congruent modulo $n$ to $y$, denoted by $x \equiv y \pmod{n}$ provided $n \mid y - x$.

Theorem
Let $n$ be a positive integer. The relation
\[
\{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{n}\}
\]
is an equivalence relation.

Definition
Let $\Omega$ be a set and let $\mathcal{P}$ be its power set. Given $A, B \in \mathcal{P}$, define a relation $R$ on $\mathcal{P}$ by $(A, B) \in R$ provided that $|A| = |B|$. In other words, $A$ and $B$ are related provided that they have the same size. This is called the has-the-same-size-as relation.

Theorem
The has-the-same-size-as relation is an equivalence relation.
Definition
Let $R$ be an equivalence relation on a set $A$ and let $a \in A$. The equivalence class of $a$, denoted by $[a]$, is the set of all elements of $A$ that are related to $a$ through the relation $R$; that is,

$$[a] = \{ x \in A : xRa \}$$

Problem
Finish the theorem: $x \in [a]$ if and only if . . .

Problem
For the congruence modulo 3 relation, what is the equivalence class of 0? of 1? of 2? of 3?

Problem
For the has-the-same-size-as relation on the power set of $\{a, b, c, d, e\}$, what is the equivalence class of $\{a, d\}$? of $\{b, c, e\}$? of $\emptyset$?
### Some theorems

**Theorem**

*Let $R$ be an equivalence relation on a set $A$ and let $a \in A$. Then $a \in [a]$.***

**Theorem**

*Let $R$ be an equivalence relation on a set $A$ and let $a, b \in A$. Then $aRb$ if and only if $[a] = [b]$.***

**Theorem**

*Let $R$ be an equivalence relation on a set $A$ and let $a, x, y \in A$. If $x, y \in [a]$, then $xRy$. (Exercise 14.9)***

**Theorem**

*Let $R$ be an equivalence relation on a set $A$ and suppose $[a] \cap [b] \neq \emptyset$. Then $[a] = [b]$.***

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**Partitioning by equivalence classes**

**Definition**

A **partition** of a set $A$ is a collection of nonempty, pairwise disjoint subsets of $A$ whose union is $A$.

**Theorem**

*Let $R$ be an equivalence relation on a set $A$. The equivalence classes of $R$ form a partition of $A$.***

**Problem**

*Partition the power set of $\{a, b, c\}$ into the equivalence classes of the has-the-same-size-as relation.*