§13 Relations

Tom Lewis

Fall Term 2010
Outline

1. Some examples
2. The inverse of a relation
3. Some properties a relation may have
Example

Here are some mothers and their daughters:

<table>
<thead>
<tr>
<th>Alice</th>
<th>Barb</th>
<th>Candace</th>
<th>Deborah</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edna</td>
<td>Felicia</td>
<td>GiGi</td>
<td>Hypatia</td>
</tr>
<tr>
<td>Ignacia</td>
<td>Janice</td>
<td></td>
<td>Klute</td>
</tr>
<tr>
<td>Linda</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can pair mothers and daughters together and create a grand set:

\[ R = \{(A, E), (A, I), (A, L), (B, F), (B, J), (C, G), (D, H), (D, K)\} \]

If we let \( M \) denote the set of mothers and \( D \) denote the set of daughters, then

\[ R \subseteq M \times D. \]

R is called a relation from \( M \) to \( D \).
Example

Let us continue with our example.

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In this case we will pair anyone from the set of daughters with any one who is their sister.

\[ S = \{(E, I), (E, L), (I, E), (I, L), (L, E), (L, I), (F, J), (J, F), (H, K), (K, H)\} \]

Notice that \( S \subseteq D \times D \). We call \( S \) a relation on \( D \).
Some examples

Example

Here is an abstract example:

\[ T = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3|(b - a)\}. \]

It would be impossible to list all of the members of \( T \), but we can see that

(0, 3), (6, -3), (4, 10), and (51, 39)

are members of \( T \).
Definition (Relation)

A **relation** is a set of ordered pairs.
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A relation is a set of ordered pairs.

Definition (Relation between sets)
Let $R$ be a relation and let $A$ and $B$ be sets. We say that $R$ is a relation on $A$ provided that $R \subseteq A \times A$, and we say $R$ is a relation from $A$ to $B$ provided that $R \subseteq A \times B$. 

Some notation
If $(a, b) \in R$, then we will write $aRb$ to indicate that $a$ is related to $b$ through $R$. 

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The inverse of a relation

Definition

Let $R$ be a relation. The inverse of $R$, denoted by $R^{-1}$, is the relation formed by reversing the order of all the ordered pairs.

Problem

Finish this sentence: If $R$ is a relation from $A$ to $B$, then $R^{-1}$ is a relation from . . .
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The inverse of a relation

Problem

Find the inverse relation of \( R = \{(A, E), (A, I), (A, L), (B, F), (B, J), (C, G), (D, H), (D, K)\}\).

Find the inverse relation of \( T = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3 \mid (b - a)\}\).
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The inverse of a relation

**Theorem**

Let $R$ be a relation. Then $(R^{-1})^{-1} = R.$
Definition

Let $R$ be a relation on a set $A$. 

Some properties a relation may have
Definition

Let \( R \) be a relation on a set \( A \).

- \( R \) is **reflexive** provided that \( xRx \) for all \( x \in A \).
Definition

Let $R$ be a relation on a set $A$.

- $R$ is **reflexive** provided that $xRx$ for all $x \in A$.
- $R$ is **irreflexive** provided that $(x, x) \notin R$ for all $x \in A$. 
Some properties a relation may have

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Let $R$ be a relation on a set $A$.

- $R$ is reflexive provided that $xRx$ for all $x \in A$.
- $R$ is irreflexive provided that $(x, x) \notin R$ for all $x \in A$.
- $R$ is symmetric provided that if $xRy$, then $yRx$ for all $x, y \in A$.
- $R$ is transitive provided that if $xRy$ and $yRz$, then $xRz$ for all $x, y, z \in A$.
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- $R$ is **antisymmetric** provided that if $xRy$ and $yRx$, then $x = y$ for all $x, y \in A$.

Question

Is irreflexive the same as not reflexive?
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Problem

Let $S$ be the relation

$$S = \{(E, I), (E, L), (I, E), (I, L), (L, E), (L, I), (F, J), (J, F), (H, K), (K, H)\}$$

Problem

Let $T$ be the relation

$$T = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3 | (b - a)\}.$$ 