§8 Factorial

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Outline

1. Factorial

2. Product notation

3. The Gamma function
**Definition**

The number of length \( n \) lists that can be created by selecting without replacement from a pool of size \( n \) is given by

\[
(n)_n = n \times (n - 1) \times \cdots \times 3 \times 2 \times 1.
\]

This quantity occurs with such frequency in mathematics that we denote it by \( n! \). By convention we set

\[
0! = 1.
\]

**Problem**

*Let \( n \) and \( k \) be nonnegative integers with \( 0 \leq k \leq n \). Show that*

\[
(n)_k = \frac{n!}{(n - k)!}
\]

**Product notation**

**Definition**

Given a sequence of real numbers \( a_1, a_2, \ldots, a_n \), let

\[
\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdots a_n.
\]

The variable \( k \) is called the index of the product. Any choice of index variable will achieve the same thing.

**Problem**

*Express \( n! \) in product notation.*
Problem

Evaluate the following products:

- \( \prod_{i=1}^{5} (3i + 2) \)
- \( \prod_{j=2}^{8} (j + 1)/j \)

Problem

Express \( 1 \times 3 \times 5 \times 9 \times 11 \times 13 \times 15 \) using...

- product notation
- factorials and powers
Definition
For each real number $x$, $x > 0$, define

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt.$$  

This is Euler’s Gamma function.

Problem
- **Evaluate** $\Gamma(1)$ and $\Gamma(2)$.
- **Show that**
  $$\Gamma(x + 1) = x\Gamma(x).$$
- **Show that** $\Gamma(m + 1) = m!$ for each integer $m \geq 0$.  

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