§12 Combinatorial Proofs

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Outline

1. What is a combinatorial identity
2. The method in action
An overview

Consider the identity: for each \( n \geq 1 \),
\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}
\]
This can be proved in any number of ways. Here is what a combinatorial proof entails: find a set \( A \) such that when counted one way gives the left-hand side and when counted another way gives the right-hand side.
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Problem

Prove that for each $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

by counting handshakes.
Problem

Show that

\[ 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1. \]

by counting subsets of a set \( A \) with \( n \) elements.
Problem

Let $n$ be a positive integer. Show

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$$