§11 Sets II
Operations

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Outline

1. Union and intersection
2. Set operations
3. The size of a union
4. Difference and symmetric difference
5. Cartesian products
Definition
Let $A$ and $B$ be sets.
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- The **intersection** of $A$ and $B$, denoted by $A \cap B$, is the set of all elements that are in $A$ and $B$. 
Problem

Express $A \cup B$ and $A \cap B$ using set-builder notation and the logical symbols $\lor$ and $\land$. 
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Express \( A \cup B \) and \( A \cap B \) using set-builder notation and the logical symbols \( \lor \) and \( \land \).

Problem

Express \( A \cup B \) and \( A \cap B \) using Venn diagrams.
Theorem

Let $A$, $B$, and $C$ be sets. The following are true:

Commutative

$A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative

$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

Identity

$A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$

Distributive

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Problem

Show that $(A \cap B) \cap C = (C \cap A) \cap B$.

Problem

Prove the associative property for the union of sets.
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Definition (Disjoint)

Let $A$ and $B$ be sets. We call $A$ and $B$ disjoint provided that $A \cap B = \emptyset$.

In general, let $A_1, A_2, \ldots, A_n$ be a collection of sets. The sets are called pairwise disjoint provided that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. In other words, no pair of sets has any elements in common.

Theorem (The addition principle)

Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint. Then $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.
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There are 24 students taking Math 141, 19 students taking Bio 101, and 11 students taking both Math 141 and Bio 101. How many students are taking either course?
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Theorem (Inclusion/exclusion principle)

Let $A$ and $B$ be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$
A keypad contains the digits 0 through 9. An access code consists of selecting 4 keys in succession, repetitions allowed. How many codes begin with a 3 or end with an 8?
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How many integers between 1 and 100 (inclusive) are divisible by 2 or 5?
Definition (Set difference)
Let $A$ and $B$ be sets. The difference between $A$ and $B$, denoted by $A - B$, is the set of all elements of $A$ that are not in $B$.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$
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Definition (Symmetric difference)

Let $A$ and $B$ be sets. The **symmetric difference** between $A$ and $B$, denoted by $A \Delta B$, is the set of all elements in $A$ but not $B$ or in $B$ but not $A$.

$$A \Delta B = (A - B) \cup (B - A).$$
Theorem (DeMorgan’s law)

Let $A$, $B$, and $C$ be sets. Then

$$A - (B \cup C) = (A - B) \cap (A - C) \quad \text{and} \quad A - (B \cap C) = (A - B) \cup (A - C)$$
Theorem (DeMorgan’s law)

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\[ A - (B \cup C) = (A - B) \cap (A - C) \quad \text{and} \quad A - (B \cap C) = (A - B) \cup (A - C) \]

Proof.

The proof is an exercise. Do these make sense by Venn diagrams?
Problem

An experiment consists of tossing a red die and a green die. Describe the set of outcomes of this experiment.

Definition

Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$, denoted by $A \times B$, is the set of all possible ordered pairs $(a, b)$ where $a \in A$ and $b \in B$. That is,

$$A \times B = \{ (a, b) : a \in A, b \in B \}$$
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Problem

Construct an explicit example to show that $A \times B$ is not necessarily equal to $B \times A$. 
Theorem

Let $A$ and $B$ be sets. Then

$$|A \times B| = |A| \times |B|.$$