§10 Quantifiers

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Outline

1. Overview
2. There is
3. For all
4. Negation
5. Combining quantifiers
How many?

Consider the following statements about the books in my library:

- **There is** a book in my library that I have read.
- **All** of the books in my library have an index.
- **There exists** a book in my library that has a green cover.
- **Every** book in my library is boring.
- **Most** of the books in my library are detective novels.

### Definition

The quantifying phrase **there exists** (equivalently **there is**) is denoted by the symbol \(\exists\).

### Example

The following are equivalent:

- There is a natural number that is a perfect square and strictly between 80 and 90.
  
- \(\exists x \in \mathbb{N}, x\) is a perfect square and \(80 < x < 90\).
Proving an existential statement

Consider an existential assertion of the form

$$\exists x \in A, x \text{ satisfies some property.}$$

What is required to prove such a statement? What is required to disprove such a statement?

Problem

*Prove the following theorem:*

Theorem

$$\exists x \in \mathbb{N}, x \text{ is a perfect square and } 80 < x < 90.$$
Proving a universal statement

Consider a universal assertion:

\[ \forall x \in A, \ x \text{ satisfies some property.} \]

What is required to prove such a statement? What is required to disprove such a statement?

Problem

Let \( A = \{ x \in \mathbb{Z} : 4 \mid x \} \). Prove the following theorem:

Theorem

\[ \forall x \in A, \ x \text{ is even.} \]

Negation

Problem

Form the negation of each of the following quantified statements:

- \( \forall x \in A, \ x \text{ satisfies property } P. \)
- \( \exists x \in A, \ x \text{ satisfies property } P. \)

A helpful mnemonic

\[ \neg \forall x \in A, \cdots = \exists x \in A, \neg \cdots \quad \text{and} \quad \neg \exists x \in A, \cdots = \forall x \in A, \neg \cdots \]
Problem

Let $A = \{x \in \mathbb{Z} : 4 \mid x\}$ Negate the following:

- $\forall x \in A, 6 \mid x$.
- $\exists x \in A, x$ is odd.

Combining quantifiers

Combinations of quantifiers

Statements can combine two or more quantifiers. The order in which the quantifiers are listed is important.

Problem

Let $E$ denote the set of even integers. What is the difference between these two statements

$\forall x \in E, \exists j \in \mathbb{Z}, x = 2j$

$\exists j \in \mathbb{Z}, \forall x \in E, x = 2j$
Negate the following statement:

$$\exists j \in \mathbb{Z}, \forall x \in E, x = 2j$$