§10 Quantifiers

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Outline

1. Overview
2. There is
3. For all
4. Negation
5. Combining quantifiers
How many?

Consider the following statements about the books in my library:

There is a book in my library that I have read.

All of the books in my library have an index.

There exists a book in my library that has a green cover.

Every book in my library is boring.

Most of the books in my library are detective novels.
How many?

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- **Every** book in my library is boring.
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Consider the following statements about the books in my library:

- **There is** a book in my library that I have read.
- **All** of the books in my library have an index.
- **There exists** a book in my library that has a green cover.
- **Every** book in my library is boring.
- **Most** of the books in my library are detective novels.
Definition

The quantifying phrase \textit{there exists} (equivalently \textit{there is}) is denoted by the symbol $\exists$. 

Example

The following are equivalent:

\[ \exists x \in \mathbb{N}, x \text{ is a perfect square and } 80 < x < 90. \]
Definition

The quantifying phrase *there exists* (equivalently *there is*) is denoted by the symbol $\exists$.

Example

The following are equivalent:

There is a natural number that is a perfect square and strictly between 80 and 90.

$\exists x \in \mathbb{N}, \ x \text{ is a perfect square and } 80 < x < 90.$
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Example

The following are equivalent:

- There is a natural number that is a perfect square and strictly between 80 and 90.
- $\exists x \in \mathbb{N}, x$ is a perfect square and $80 < x < 90$. 
Proving an existential statement

Consider an existential assertion of the form

$$\exists x \in A, \ x \text{ satisfies some property}.$$ 

What is required to prove such a statement? What is required to disprove such a statement?
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Problem

Prove the following theorem:
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Theorem

\[ \exists x \in \mathbb{N}, \ x \text{ is a perfect square and } 80 < x < 90. \]
Definition

The quantifying phrase for all (equivalently each, every, any) is denoted by the symbol \( \forall \).
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Example

Let $A = \{x \in \mathbb{Z} : 4|\,x\}$. The following are equivalent:
Definition

The quantifying phrase for all (equivalently each, every, any) is denoted by the symbol $\forall$.

Example

Let $A = \{x \in \mathbb{Z} : 4|\text{x}\}$.

The following are equivalent:

- Any integer that is divisible by 4 is even.
Definition

The quantifying phrase for all (equivalently each, every, any) is denoted by the symbol \( \forall \).

Example

Let \( A = \{ x \in \mathbb{Z} : 4 | x \} \). The following are equivalent:

- Any integer that is divisible by 4 is even.
- \( \forall x \in A, x \) is even.
Proving a universal statement

Consider a universal assertion:

$$\forall x \in A, \ x \text{ satisfies some property.}$$

What is required to prove such a statement? What is required to disprove such a statement?
Proving a universal statement

Consider a universal assertion:

\[ \forall x \in A, \ x \text{ satisfies some property.} \]

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Problem

Let \( A = \{ x \in \mathbb{Z} : 4 \mid x \} \). Prove the following theorem:
Proving a universal statement

Consider a universal assertion:

\[ \forall x \in A, \ x \text{ satisfies some property.} \]

What is required to prove such a statement? What is required to disprove such a statement?

Problem

Let \( A = \{ x \in \mathbb{Z} : 4 | x \} \). Prove the following theorem:

Theorem

\( \forall x \in A, \ x \text{ is even.} \)
Problem

Form the negation of each of the following quantified statements:
Problem

Form the negation of each of the following quantified statements:

- $\forall x \in A, x$ satisfies property $P$. 
Problem

Form the negation of each of the following quantified statements:

- $\forall x \in A, x$ satisfies property $P$.
- $\exists x \in A, x$ satisfies property $P$. 
Problem

Form the negation of each of the following quantified statements:

- \( \forall x \in A, x \text{ satisfies property } P. \)
- \( \exists x \in A, x \text{ satisfies property } P. \)

A helpful mnemonic

- \( \neg \forall x \in A, \cdots = \exists x \in A, \neg \cdots \quad \text{and} \quad \neg \exists x \in A, \cdots = \forall x \in A, \neg \cdots \)
Let $A = \{x \in \mathbb{Z} : 4 | x\}$ Negate the following:

$\forall x \in A, 6 \nmid x.$

$\exists x \in A, \text{x is odd}.$
Problem

Let \( A = \{ x \in \mathbb{Z} : 4 \mid x \} \) Negate the following:

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- $\exists x \in A, x \text{ is odd.}$
Combinations of quantifiers

Statements can combine two or more quantifiers. The order in which the quantifiers are listed is important.

Problem

Let $E$ denote the set of even integers. What is the difference between these two statements?

1. $\forall x \in E, \exists j \in \mathbb{Z}, x = 2j$
2. $\exists j \in \mathbb{Z}, \forall x \in E, x = 2j$
Combining quantifiers

Statements can combine two or more quantifiers. The order in which the quantifiers are listed is important.

Problem

Let $E$ denote the set of even integers. What is the difference between these two statements

\[
\forall x \in E, \exists j \in \mathbb{Z}, x = 2j \\
\exists j \in \mathbb{Z}, \forall x \in E, x = 2j
\]
Problem

Negate the following statement:

\[ \exists j \in \mathbb{Z}, \forall x \in E, x = 2j \]