§3 Theorem

Tom Lewis

Fall Term 2010
Outline

1. Statements

2. Practical ‘proof’ versus mathematical proof

3. Connectives and inferences

4. Some if/then terminology
Definition (Statement)

In logic a statement is a declarative sentence that is either true or false.

Aristotle is a mortal.

How old are you?

The population of Greenville is greater than the population of South Carolina.
Definition (Statement)
In logic a **statement** is a declarative sentence that is either true or false.

Problem
*Which of the following are statements?*
Definition (Statement)

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Problem

Which of the following are statements?

- Aristotle is a mortal.
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Problem

*Which of the following are statements?*

- Aristotle is a mortal.
- How old are you?
- The population of Greenville is greater than the population of South Carolina.
Mathematics and certainty

A mathematical statement can be either true or false, but this is not entirely helpful, since we may not know for certain which camp it is in.

Goldbach's Conjecture:

Every even integer greater than 2 can be expressed as the sum of two primes.

Try out some small cases.

Is this statement true? If yes, how do we know? If not, why not?
Mathematics and certainty

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- Try out some small cases.
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Goldbach

Every even integer greater than 2 can be expressed as the sum of two primes.

- Try out some small cases.
- Is this statement true? If yes, how do we know? If not, why not?
Some terminology about mathematical statements

Here are some helpful terms:

- **Theorem**: A true declarative statement about mathematics for which there is a proof.
- **Conjecture**: A declarative statement about mathematics whose truth has not been ascertained.
- **False**: A mathematical statement is false provided that we can demonstrate that it is false.

**Problem**

How should we categorize the following statements?

1. The difference between consecutive perfect squares is an odd number.
2. Each odd number greater than 2 is a prime.
Some terminology about mathematical statements

Here are some helpful terms:

- A **theorem** is a true declarative statement about mathematics for which there is a proof.

Problem

How should we categorize the following statements?

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We will say that a mathematical statement is false provided that we can demonstrate that it is false.

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Problem

*How should we categorize the following statements?*

- *The difference between consecutive perfect squares is an odd number.*
Some terminology about mathematical statements

Here are some helpful terms:

- A **theorem** is a true declarative statement about mathematics for which there is a proof.
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Problem

*How should we categorize the following statements?*

- The difference between consecutive perfect squares is an odd number.
- Each odd number greater than 2 is a prime.
Practical and Mathematical Proof

The area of a circle is the same as the area of a right triangle whose base is the circumference of the circle and whose height is the radius of the circle.
Minding your $p$’s and $q$’s

Throughout this discussion, let $p$ and $q$ be statements. For the purpose of considering examples, we will let $p$ and $q$ stand for the specific statements:

$p :$ Ann was late for work.

$q :$ Ann was fired.
Negation

If the statement \( p \) is true, then \( \neg p \) is false. Likewise if \( p \) is false, then \( \neg p \) is true. This obvious relationship can be encoded in a truth table.
Negation

If the statement \( p \) is true, then \( \neg p \) is false. Likewise if \( p \) is false, then \( \neg p \) is true. This obvious relationship can be encoded in a truth table.

The truth table for \( \neg \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
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<tbody>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>T</td>
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</table>
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<tbody>
<tr>
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<td>$F$</td>
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</table>

Rules of inference
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<tr>
<td>$F$</td>
<td>$T$</td>
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</tbody>
</table>

Rules of inference

Rule 1 Either $p$ is true or $\neg p$ is true, but not both.
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If the statement $p$ is true, then $\neg p$ is false. Likewise if $p$ is false, then $\neg p$ is true. This obvious relationship can be encoded in a truth table.

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</tbody>
</table>

Rules of inference

Rule 1  Either $p$ is true or $\neg p$ is true, but not both.

Rule 2  It is not possible for $p$ and $\neg p$ to be true at the same time.
Connectives and inferences

Connecting with \textit{and}

\textbf{Example}

Consider the statement: \textit{Ann was late for work and Ann was fired}. In propositional variables, we would write this using the connective $\land$:

$$ p \land q $$
Connecting with *and*

**Example**
Consider the statement: *Ann was late for work and Ann was fired.* In propositional variables, we would write this using the connective $\land$:

$$p \land q$$

**Problem**

*In what case(s) will the statement $p \land q$ be true?*
Truth table for ∧

Here is the truth table for ∧ (and):

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∧ q</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
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</table>
**Truth table for \( \land \)**

Here is the truth table for \( \land \) (and):

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<thead>
<tr>
<th></th>
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<th>( p ) ( \land q )</th>
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<tbody>
<tr>
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</tbody>
</table>

**Some rules of inference**
Truth table for $\land$

Here is the truth table for $\land$ (and):

<table>
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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
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<tbody>
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</tbody>
</table>

Some rules of inference

Rule 3 If $p \land q$ is true, then both $p$ and $q$ must be true.
Truth table for $\land$  
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<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
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<td>$F$</td>
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Some rules of inference

Rule 3 If $p \land q$ is true, then both $p$ and $q$ must be true.

Rule 4 If $p \land q$ is false, then at least one of $p$ or $q$ is false.
Connecting with or

Example

Consider the statement: *Ann was late for work or Ann was fired.* In propositional variables, we would write this using the connective $\lor$:

$$p \lor q$$
Connecting with *or*

**Example**

Consider the statement: *Ann was late for work or Ann was fired.* In propositional variables, we would write this using the connective \( \lor \):

\[ p \lor q \]

**Problem**

*In what case(s) will the statement* \( p \lor q \) *be true?*
## Truth table

Here is the truth table for $\lor$ (or):

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
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</thead>
<tbody>
<tr>
<td>F</td>
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</table>
**Truth table**

Here is the truth table for \( \lor \) (or):

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 T & F & T \\
 T & T & T \\
\end{array}
\]

**Some rules of inference**
Truth table

Here is the truth table for $\lor$ (or):

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<tr>
<th>$p$</th>
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<th>$p \lor q$</th>
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Some rules of inference

Rule 5 If $p \lor q$ is false, then both $p$ and $q$ must be false.
**Truth table**

Here is the truth table for \( \lor \) (or):

<table>
<thead>
<tr>
<th></th>
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<th>( p \lor q )</th>
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**Some rules of inference**

**Rule 5** If \( p \lor q \) is false, then both \( p \) and \( q \) must be false.

**Rule 6** If \( p \lor q \) is true, then at least one of \( p \) or \( q \) is true.
An application of Rule 6

Consider the following inference: Ann was late for work or Ann was fired. But Ann was not late for work; therefore, Ann was fired.
An application of Rule 6

Consider the following inference: Ann was late for work or Ann was fired. But Ann was not late for work; therefore, Ann was fired.

If \( p \lor q \) is true, then by Rule 6 we know that at least one \( p \) or \( q \) (possibly both) must be true. However, since \( p \) is false, \( q \) must be true.
Example

Consider the statement: *If Ann was late for work, then Ann was fired.* In propositional variables, we would write this using the connective $\rightarrow$:

$$p \rightarrow q$$
Connecting with the if/then conditional

Example
Consider the statement: *If Ann was late for work, then Ann was fired.* In propositional variables, we would write this using the connective $\rightarrow$:

$$p \rightarrow q$$

Problem
*In what case(s) will the statement $p \rightarrow q$ be true?*
Truth table

Here is the truth table for $\rightarrow$ (if/then):

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<tr>
<th>$p$</th>
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Some rules of inference

Rule 7
If $p \rightarrow q$ is true, and $p$ is true, then $q$ must be true. (Modus Ponens)

Rule 8
If $p \rightarrow q$ is true, and $q$ is false, then $p$ must be false. (Modus Tollens)

Rule 9
If $p \rightarrow q$ is false, then $p$ must be true and $q$ must be false.
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Rule 9  If $p \rightarrow q$ is false, then $p$ must be true and $q$ must be false.
Problem

Is the following statement true or false? If an integer is both a perfect square and prime, then it is negative.
Problem

Is the following statement true or false? If an integer is both a perfect square and prime, then it is negative.

Definition (Vacuous Truth)

Statements of the form If $p$, then $q$ in which the condition $p$ is impossible are called vacuous. Such statements are true because they have no exceptions!
Example

Consider the statement: *Ann was late for work if and only if Ann was fired.* In propositional variables, we would write this using the connective ↔:

\[ p \leftrightarrow q \]

The biconditional is really two conditional statements connected by an and and:

\[ p \leftrightarrow q \text{ is equivalent to } (p \rightarrow q) \land (q \rightarrow p). \]

Connectives and inferences
Example

Consider the statement: *Ann was late for work if and only if Ann was fired.* In propositional variables, we would write this using the connective $\leftrightarrow$:

$$p \leftrightarrow q$$

The biconditional is really two conditional statements connected by an and:

$$p \leftrightarrow q \text{ is equivalent to } (p \rightarrow q) \land (q \rightarrow p).$$

Problem

*In what case(s) will the statement $p \leftrightarrow q$ be true?*
### Truth table

Here is the truth table for $\leftrightarrow$ (if and only if):

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**Some rules of inference**
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### Some rules of inference

**Rule 11** If $p \leftrightarrow q$ is true, then $p$ and $q$ have the same truth value.
Connectives and inferences

Truth table

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Some rules of inference

Rule 11  If $p \leftrightarrow q$ is true, then $p$ and $q$ have the same truth value.
Rule 12  If $p \leftrightarrow q$ is false, then $p$ and $q$ have opposite truth values.
Terms of use

There are numerous ways to express the statement “If \( p \), then \( q \).” Here are a few such expressions. Try to unlock the sense of each of them.

- \( p \) implies \( q \).
- \( p \) is sufficient for \( q \) or \( p \) is a sufficient condition for \( q \).
- \( q \) is necessary for \( p \).
- \( p \), only if \( q \).
Terms of use

There are numerous ways to express the statement “If $p$, then $q$.” Here are a few such expressions. Try to unlock the sense of each of them.

- “$p$ implies $q$.”

- “$p$ is sufficient for $q$.” or “$p$ is a sufficient condition for $q$.”

- “$q$ is necessary for $p$.”

- “$p$, only if $q$.”
Some if/then terminology

Terms of use

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Some if/then terminology

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- “$p$, only if $q$."

Tom Lewis ()

§3 Theorem

Fall Term 2010 20 / 20