54

15



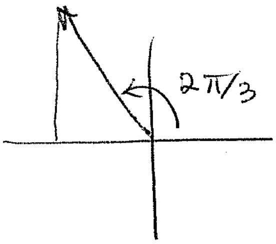
**Test #1** Name Answer key.

**Directions**: Show all of your work. Use proper notation.

1. Convert the point $(2, 2\pi/3, -2)$ from cylindrical to Cartesian coordinates.

5

$x = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right) = -1$




$y = 2\sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}.$

$z = -2$

$(1, \sqrt{3}, -2)$

2. Translate the equation $\rho = 2\cos(\varphi)$ from spherical to Cartesian coordinates and identify the surface.

5

$\rho^2 = 2\rho\cos\varphi \Rightarrow x^2 + y^2 + z^2 = 2z$

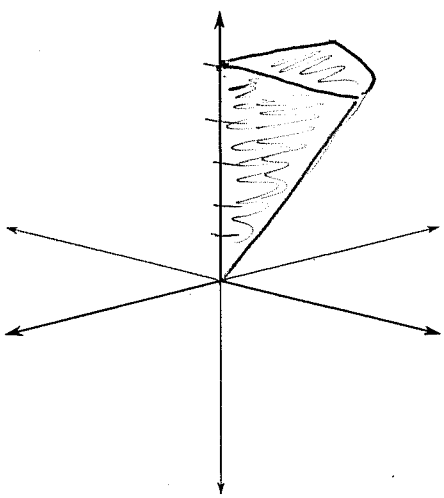
$x^2 + y^2 + z^2 - 2z + 1 = 1$

$x^2 + y^2 + (z-1)^2 = 1$

$C = (0, 0, 1)$ ; $r = 1$ sphere of radius 1 centered @ $(0, 0, 1)$.

3. Sketch the solid whose cylindrical coordinates satisfy the inequalities $0 \le r \le z \le 5$ and $\frac{\pi}{2} \le \theta \le \pi$.

5



This is a ¼ th cone, topped-off @ $z = 5$

12



4. Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$.

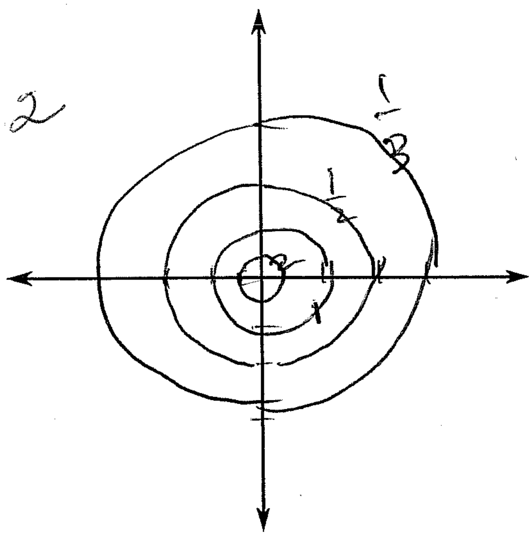
(a) What is the domain of $f$?

2 $D = \{(x,y) : (x,y) \neq (0,0)\}$

(b) Sketch the level curves corresponding to the levels $c = 1/3, 1/2, 1, 2$. Label the level curves accordingly.

$\dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{3}$, $x^2+y^2 = 3^2$

2




(c) What is the range of $f$? $(0, \infty)$.

2

5. Evaluate the following limits:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$\displaystyle\lim_{(x,0)} \frac{0}{\sqrt{x^2+0^2}} = 0$

$\displaystyle\lim_{(x,x)} \frac{x^2}{(2)x^2} = \frac{1}{2}$

since $0 \neq \frac{1}{2}$, the limit does not exist.

3

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+x^2+2y^2x^2+2y^2}{x^2+2y^2}$

$= \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2(x^2+1)+2y^2(x^2+1)}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+1)\cancel{(x^2+2y^2)}}{\cancel{x^2+2y^2}}$

$= 1.$

3

12



6. Work each of the following:

(a) Let $f(x,y) = x^3 - 3xy^2$. Show that $f$ satisfies Laplace's equation, $f_{xx} + f_{yy} = 0$.

3

$$f_x = 3x^2 - 3y^2 \qquad f_y = -6xy$$

$$f_{xx} = 6x \qquad f_{yy} = -6x$$

$$\therefore f_{xx} + f_{yy} = 6x - 6x = 0.$$

(b) Find all second-order partial derivatives of $f(x,y) = e^{xy}$.

$$f_x = e^{xy}(y) \qquad f_y = e^{xy}x$$

3

$$\boxed{\begin{aligned} f_{xy} &= e^{xy}(xy) + e^{xy}(1) \qquad & f_{yx} &= e^{xy}xy + e^{xy} \\ f_{xx} &= e^{xy}(y^2) + e^{xy}(0) & f_{yy} &= e^{xy}(x^2) + e^{xy}(0) \end{aligned}}$$

(c) Let $f(x,y,z) = (xyz^2, 2x - 3y + z^2)$. Find $(Df)(1,0,2)$.

3

$$Df = \begin{array}{c} \\ xyz^2 \\ 2x-3y+z^2 \end{array} \begin{array}{c} x \qquad\quad y \qquad\quad z \\ \begin{bmatrix} yz^2 & xz^2 & 2xyz \\ 2 & -3 & 2z \end{bmatrix} \end{array}$$

$$Df(1,0,2) = \begin{bmatrix} 0 & 4 & 0 \\ 2 & -3 & 4 \end{bmatrix}$$

(d) Let $f(x,y,z) = \dfrac{x+y}{e^z}$ and let $a = (2,-1,0)$. Find $\nabla f(a)$.

$$f(x,y,z) = xe^{-z} + ye^{-z}$$

2.

$$\nabla f = \left(e^{-z},\ e^{-z},\ [xe^{-z} + ye^{-z}](-1)\right)$$

$$\nabla f(2,-1,0) = \left(e^0,\ e^0,\ (2e^0 - 1e^0)(-1)\right)$$

$$= (1, 1, -1).$$

15



7. Let $f(x,y) = 2x + 3y + 8$. Show that if $\|(x,y) - (2,3)\| < \delta$, then $|f(x,y) - 21| < 5\delta$.

$\|(x,y) - (2,3)\| < \delta \Rightarrow |x-2| < \delta$ and $|y-3| < \delta$.

5 Thus $|f(x,y) - 21| = |2x+3y+8-21|$

$= |2x+3y-13| = |2(x-2)+3(y-3)| \leq 2|x-2| + 3|y-3|$
$< 2\delta + 3\delta = 5\delta$.

8. Let $f(x,y,z)$ be a differentiable function, let $a = (1,3,2)$, and let

5

$$f(a) = 5, \quad \frac{\partial f}{\partial x}(a) = -1, \quad \frac{\partial f}{\partial y}(a) = .5, \quad \text{and} \quad \frac{\partial f}{\partial z}(a) = .3$$

Use this information to give a reasonable approximation to the value of $f(1.1, 2.9, 2.2)$.

$h(a) = 5 + (-1)(x-1) + (.5)(y-3) + (.3)(z-2)$

$h(1.1, 2.9, 2.2) = 5 + (-1)(.1) + (.5)(-.1) + (.3)(.2)$
$= 5 - .1 - .05 + .06 = 4.91$

$f(1.1, 2.9, 2.2) \approx 4.91$.

9. Let $f(x,y) = 8 + 3x + 4y + xy$.

(a) Find the equation of a tangent plane $z = h(x,y)$ to the surface $z = f(x,y)$ at $(0,0)$.

$f(0,0) = 8$

3

$\nabla f(x,y) = (3, 4)$

$z = h(x,y) = 8 + (3,4)(x-0, y-0) = 8 + 3x + 4y$

(b) Show that $f$ is differentiable at $(0,0)$ by showing that

2

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\sqrt{x^2+y^2}} = 0.$$

$\lim_{(x,y)\to(0,0)} \dfrac{8+3x+4y+xy - (8+3x+4y)}{\sqrt{x^2+y^2}} =$

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \dfrac{r^2 \sin\theta \cos\theta}{r}$

$= 0.$