Assignment #9 **Name** Answer Key

Due 20 May 2008

1. Let C be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ oriented counter-clockwise. Evaluate the integral $\int_C (-ydx + xdy)$ by the following methods:

(a) By direct calculation of the path integral.

$$x(t) = (2\cos t, 3\sin t), \quad 0 \le t \le 2\pi$$

$$x'(t) = (-2\sin t, 3\cos t)$$

$$F(x, y) = (-y, x)$$

$$F(x(t)) = (-3\sin t, 2\cos t)$$

$$F(x(t)) \cdot x'(t) = 6(\sin^2 t + \cos^2 t) = 6$$

$$\int_C F \cdot ds = \int_0^{2\pi} 6 \cdot dt = 6(2\pi) = 12\pi$$

(b) Using Green's Theorem.

$$\int_C F \cdot ds = \iint_D (\operatorname{curl} F) \cdot \hat{k} \, dA$$

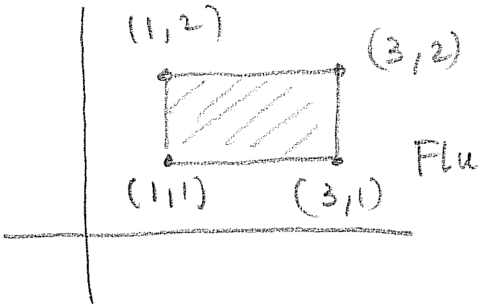
But $(\operatorname{curl} F) \cdot \hat{k} = 2$ $\therefore$

$$\int_C F \cdot ds = \iint_D 2 \, dA = 2 \operatorname{area}(D)$$

$$= 2(\pi \cdot 2 \cdot 3)$$

$$= 12\pi$$

2. Let a vector field be given by $F = (xy, y + x^2)$ and let D be the rectangle in the plane with corners at $(1,1)$, $(3,1)$, $(3,2)$, and $(1,2)$. Calculate the flux of F across the boundary of D.



By the Divergence Thm,

$$\text{Flux} = \int_{\partial D} F\cdot n \, ds = \iint_D (\operatorname{div} F)\, dA$$

But $\operatorname{div} F = y + 1 \quad \therefore \quad \text{Flux} = \int_{x=1}^{3}\int_{y=1}^{2} (y+1)\, dy\, dx$

$$= \int_1^3 \left(\tfrac{1}{2}y^2 + y \Big|_1^2\right) dx = \int_1^3 \frac{5}{2}\, dx = \left(\frac{5}{2}\right)(2) = 5$$

3. Find the work done by the force $F = (-yz, -xz, -xy)$ when a particle moves on the straight line segment joining $(1,1,1)$ to $(3,4,2)$.

$X(t) = (1,1,1) + t(2,3,1) = (1+2t, 1+3t, 1+t),$

$0 \le t \le 1$

$X'(t) = (2,3,1).$

$F(X(t)) = \left(-(1+3t)(1+t), -(1+2t)(1+t), -(1+2t)(1+3t)\right)$

$= -\left(1+4t+3t^2, 1+3t+2t^2, 1+5t+6t^2\right)$

$F(X(t))\cdot X'(t) = -\left\{(1+4t+3t^2)(2) + (1+3t+2t^2)(3) + (1+5t+6t^2)\right\}$

$= -\left\{6 + 22t + 18t^2\right\}$

$$\text{Work} = \int_X F\cdot ds = -\int_0^1 (6 + 22t + 18t^2)\, dt$$

$$= -(6t + 11t^2 + 6t^3)\Big|_0^1 = -23.$$

This is easier using a potential function!!