1 bonus point

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Assignment #3 **Name** Answer Key.

Due 31 March 2008

1. Let $f : \mathbb{R} \to \mathbb{R}$ have a continuous second derivative and let $c > 0$ be a constant. Let $w(x, t) = f(x - ct)$ and show that
$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$
which is the one-dimensional *wave equation*. Hint: let $u = x - ct$ and make a dependency chart.

$$\begin{matrix} x \searrow & & \\ & u \longrightarrow & w = f(u). \\ t \nearrow & & \end{matrix}$$

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$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} = f'(u)(-c). \qquad \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}(f'(u))(-c)$$
$$= f''(u)(-c)^2 = c^2 f''(u)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} = f'(u)(1) \qquad \frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(f'(u)) = f''(u)(1)$$
$$= f''(u).$$

Thus $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$, as was to be shown.

2. Let $x : \mathbb{R}^3 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ according to the formulas
$$x(t_1, t_2, t_3) = \begin{bmatrix} t_1 t_2 t_3^2 \\ t_1^2 + t_2^2 + t_3^2 \end{bmatrix} \quad \text{and} \quad f(x_1, x_2) = \begin{bmatrix} 3x_1 + 4x_2 + x_1 x_2 \\ x_1 x_2 + x_1^3 + x_2^2 \end{bmatrix}$$
Use the chain rule to find $D(f \circ x)(1, 0, 1)$.

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$$\mathbb{R}^3 \xrightarrow{x} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$$
$$(1,0,1) \longrightarrow (0,2) \quad \therefore \quad D(f\circ x)(1,0,1) = Df(0,2)\cdot Dx(1,0,1)$$

$$Df = \begin{bmatrix} 3 + x_2 & 4 + x_1 \\ x_2 + 3x_1^2 & x_1 + 2x_2 \end{bmatrix} \qquad Df(0,2) = \begin{bmatrix} 5 & 4 \\ 2 & 4 \end{bmatrix}$$

$$Dx = \begin{bmatrix} t_2 t_3^2 & t_1 t_3^2 & 3t_1 t_2 t_3^2 \\ 2t_1 & 2t_2 & 2t_3 \end{bmatrix}; \quad Dx(1,0,1) = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$D(f\circ x)(1,0,1) = \begin{bmatrix} 5 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 8 \\ 8 & 2 & 8 \end{bmatrix}$$

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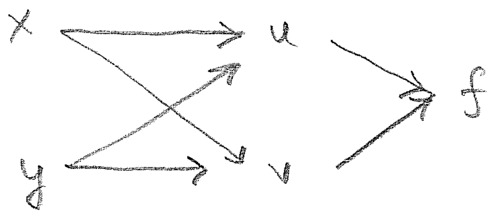
3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 function and let θ be a fixed angle. Let $u = x\cos(\theta) - y\sin(\theta)$ and $v = x\sin(\theta) + y\cos(\theta)$. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$$



$$f_x = f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = f_u \cos\theta + f_v \sin\theta.$$

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}\left(f_u \cos\theta + f_v \sin\theta\right) \\ &= \frac{\partial}{\partial x}(f_u)\cos\theta + f_u \frac{\partial}{\partial x}\cos\theta + \frac{\partial}{\partial x} f_v \sin\theta + f_v \frac{\partial}{\partial x}\sin\theta \\ &= \left(f_{uu}\cos\theta + f_{uv}\sin\theta\right)\cos\theta + 0 + \left(f_{vu}\cos\theta + f_{vv}\sin\theta\right)\sin\theta \\ &\quad + 0 \\ &= f_{uu}\cos^2\theta + 2 f_{uv}\sin\theta\cos\theta + f_{vv}\sin^2\theta. \end{aligned}$$

Likewise

$$f_y = f_u(-\sin\theta) + f_v(\cos\theta)$$

$$\begin{aligned} f_{yy} &= \left(f_{uu}(-\sin\theta) + f_{uv}\cos\theta\right)(-\sin\theta) \\ &\quad + \left(f_{vu}(-\sin\theta) + f_{vv}\cos\theta\right)\cos\theta. \\ &= f_{uu}\sin^2\theta - 2 f_{uv}\sin\theta\cos\theta + f_{vv}\cos^2\theta. \end{aligned}$$

Thus

$$\begin{aligned} f_{xx} + f_{yy} &= f_{uu}\left(\sin^2\theta + \cos^2\theta\right) + \\ &\qquad f_{vv}\left(\sin^2\theta + \cos^2\theta\right) \\ &= f_{uu} + f_{vv} \qquad \text{Q.E.D.} \end{aligned}$$