§7.4–Partial Fractions

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Outline

The method illustrated

Terminology

Factoring Polynomials

Partial fraction decompositions

Further examples
Problem

Let

\[ f(x) = \frac{x + 1}{2x^2 + 7x + 6}. \]

1. Show that

\[ f(x) = \frac{1}{x + 2} - \frac{1}{2x + 3}. \]

The expression on the right is called the partial fraction decomposition.

2. Use this decomposition to evaluate \( \int f(x)\,dx \).

Definition

A rational function is any function of the form \( f(x) = P(x)/Q(x) \), where \( P \) and \( Q \) are polynomials. The rational function \( f \) is said to be proper if \( \deg P < \deg Q \).

Example

Here are two examples:

- \( f(x) = \frac{x + 1}{2x^2 + 7x + 6} \) is proper.
- \( g(x) = \frac{x^3 + 2x + 4}{x^2 - 1} \) is not proper.
Long division

Using long division, an improper rational function can be written as a sum of a polynomial and a proper rational function.

Problem

Express

\[
\frac{x^4 - 4x^2 + 3x + 4}{x^2 - 4}
\]

as the sum of a polynomial plus a proper rational function.
Definition (Types of factors)

There are two types of factors:

1. A factor of the form $Ax + B$ is called linear.
2. A factor of the form $Ax^2 + Bx + C$ for which $B^2 - 4AC < 0$ is called an irreducible quadratic.

Check!
Is $2x^2 - x - 36$ irreducible?

Problem

1. Factor $P(x) = x^3 + 2x^2 - 4x - 8$. Identify the factors and their multiplicities.
2. Factor $Q(x) = x^4 - 1$. Identify the factors and their multiplicities.
Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be expressed as a product of powers of linear factors \((ax + b)^m\) and powers of irreducible quadratic factors \((ax^2 + bx + c)^n\).

Theorem (Partial Fraction Decompositions)

Assume that the rational function \(\frac{P(x)}{Q(x)}\) is proper.

- Each factor of \(Q\) will generate terms of the partial fraction decomposition of \(P/Q\).
- To each linear factor \((ax + b)^m\) of \(Q\), the decomposition of \(P/Q\) will contain the terms

\[
\frac{D_1}{(ax + b)^1} + \cdots + \frac{D_m}{(ax + b)^m}.
\]

- To each irreducible quadratic factor \((ax^2 + bx + c)^n\) of \(Q\), the decomposition will contain the terms

\[
\frac{E_1x + F_1}{(ax^2 + bx + c)^1} + \cdots + \frac{E_nx + F_n}{(ax^2 + bx + c)^n}.
\]
Problem

Find the partial fraction decomposition of

\[ f(x) = (x + 1)/(2x^2 + 7x + 6) \text{ and evaluate } \int f(x) \, dx. \]

Problem

Find the partial fraction decomposition of

\[ f(x) = (3x^2 + 3x - 2)/(x^3 + 2x^2 - 4x - 8) \text{ and evaluate } \int f(x) \, dx. \]
Problem

Evaluate \[ \int \frac{x^6 + 2x^4 + x^3 - 2x^2 - x - 5}{x^4 - 1} \, dx. \]