§6.6–The Inverse Trigonometric Functions

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Outline

The inverse sine function
The inverse cosine function
The inverse tangent function
The other inverse trig functions
Miscellaneous problems
Integrals
Definition

- The sine function is one-to-one on \([-\pi/2, \pi/2]\) and has range \([-1, 1]\) on this domain.
- We define \(\sin^{-1}\) to be the inverse of sine on this domain. It follows that \(\sin^{-1}\) has domain \([-1, 1]\) and range \([-\pi/2, \pi/2]\).

Cancellation equations

Because of these restrictions, we must be a little careful with the inverse relationships:

\[
\sin (\sin^{-1}(x)) = x, \quad -1 \leq x \leq 1
\]
\[
\sin^{-1} (\sin(x)) = x, \quad -\pi/2 \leq x \leq \pi/2
\]
Problem

Evaluate the following:

- \( \sin(\sin^{-1}(0.3)) \)
- \( \sin^{-1}(\sin(14\pi/3)) \). *(Be careful. Try this on a calculator first.)*

Problem

*Show that* \( \cos(\sin^{-1}(x)) = \sqrt{1-x^2} \).
Problem

Find \( \cos \left( 2 \sin^{-1} \left( \frac{1}{4} \right) \right) \).

Theorem

\[
D_x \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.
\]
Problem

Find the derivative of $y = x \sin^{-1}(x^2)$.

Definition

- The cosine function is one-to-one on the interval $[0, \pi]$ and has range $[-1, 1]$ on that domain.
- Let $\cos^{-1}$ denote the inverse of the cosine function restricted to the domain $[0, \pi]$. Thus the domain of $\cos^{-1}$ is $[-1, 1]$ and its range is $[0, \pi]$. 
The cancellation equations

\[
\cos \left( \cos^{-1}(x) \right) = x, \quad -1 \leq x \leq 1
\]
\[
\cos^{-1} \left( \cos(x) \right) = x, \quad 0 \leq x \leq \pi
\]

Problem

- Evaluate \( \cos^{-1} \left( \cos(14\pi/3) \right) \)
- Show that \( \sin \left( \cos^{-1}(x) \right) = \sqrt{1 - x^2} \)
Theorem

\[ D_x \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}} \]

Definition

- The tangent is one-to-one on the interval \((-\pi/2, \pi/2)\) and has range \((-\infty, \infty)\) on this domain.
- Let \(\tan^{-1}\) be the inverse of the tangent function on this restricted domain. Thus the domain of \(\tan^{-1}\) is \((-\infty, \infty)\) and its range is \((-\pi/2, \pi/2)\).
The cancellation equations
\[
\tan\left(\tan^{-1}(x)\right) = x \quad -\infty < x < \infty \\
\tan^{-1}\left(\tan(x)\right) = x \quad -\pi/2 < x < \pi/2.
\]

Problem

Show that \( \sec^2\left(\tan^{-1}(x)\right) = 1 + x^2 \)
Theorem

\[ D_x \tan^{-1}(x) = \frac{1}{1 + x^2}, \quad x \in \mathbb{R}. \]
Problem

*Find* $y'$ *in each case:*

- $y = \tan^{-1}(e^x)$
- $y = \sqrt{1 - x^2} \sin^{-1}(x)$
- $y = \sin^{-1}(x) + \cos^{-1}(x)$

Basic integration formulas

- $\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1}(x) + C$
- $\int \frac{1}{1 + x^2} \, dx = \tan^{-1}(x) + C$

Problem

*Why is there no formula involving* $\cos^{-1}(x)$?
Problem

Evaluate the following integrals:

1. \( \int \frac{\tan^{-1}(x)}{1 + x^2} \, dx \)
2. \( \int_{0}^{\pi/2} \frac{\sin(x)}{1 + \cos^2(x)} \, dx \)
3. \( \int_{0}^{1} \frac{1}{\sqrt{4 - t^2}} \, dt \)
4. \( \int \frac{1}{a^2 + x^2} \, dx \), where \( a \) is any real number.