§6.2*–The Natural Logarithm Function

Tom Lewis

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Outline

The log base 10
The natural logarithm
Further properties of the natural logarithm
The graph of the log function
The number $e$, Euler's number
Further derivative problems
Integration
Logarithmic differentiation
Problem

1. \(10^x = 100\)
2. \(10^x = 1/1000\)
3. \(10^x = -2\)
4. \(10^x = 5.386\)

Properties of the base 10 logarithm function

Here are some familiar properties of the base 10 logarithm function.

- \(\log_{10}(1) = 0, \log_{10}(10) = 1, \log_{10}(100) = 2\).
- \(\log_{10}(ab) = \log_{10}(a) + \log_{10}(b), a, b > 0\).
- \(\log_{10}(1/a) = -\log_{10}(a), a > 0\).
- \(\log_{10}(a/b) = \log_{10}(a) - \log_{10}(b)\).
- \(\log_{10}(a^r) = r \log_{10}(a), a > 0, r \text{ rational}\).
Definition (The natural logarithm function)

For $x > 0$, let

$$\ln(x) = \int_1^x \frac{1}{t} \, dt.$$ 

This function is called the **natural logarithm**.

Theorem (Elementary properties of $\ln$)

- $\ln(1) = 0$
- $\ln(x) < 0$ if $0 < x < 1$
- $\ln(x) > 0$ if $x > 1$
- $D_x \ln(x) = 1/x$. *In particular, $\ln(x)$ is an increasing function.*
Problem

*Show that $0.5 \leq \ln(2) \leq 1$.\n
Theorem

- $\ln(ab) = \ln(a) + \ln(b)$, $a, b > 0$.
- $\ln(a/b) = \ln(a) - \ln(b)$.
- $\ln(a^p) = p\ln(a)$, for $p > 0$, $p$ rational. *(Homework)*
Theorem

- \( \ln \) is increasing and concave down.
- As \( x \to +\infty \), \( \ln(x) \to +\infty \).
- As \( x \to 0^+ \), \( \ln(x) \to -\infty \).

Problem

Sketch the graph of \( \ln(x) \).

Definition

- The range of \( \ln(x) \) is \(( -\infty, \infty ) \). Since \( \ln(x) \) is increasing, there exists a unique number \( e \) such that
  \[
  \ln(e) = 1.
  \]
- The number \( e \) is called Euler’s number. Note that
  \[
  e \approx 2.71828
  \]
- Since \( \ln(e) = 1 \), \( e \) is called the base of the natural logarithm function.
Theorem (The chain rule)

If \( f \) is a positive, differentiable function, then

\[
\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} f'(x).
\]

Problem

Find \( \frac{dy}{dx} \) in each case:

- \( y = \ln(x^2) \)
- \( y = x^2(\ln(x^2 + 1))^3 \)
- \( y = \ln(|x|) \)
Theorem

\[ \int \frac{1}{x} \, dx = \ln(|x|) + C. \]

Problem

Evaluate the following integrals:

- \[ \int \frac{x^2}{x^3 + 1} \, dx \]
- \[ \int_{-2}^{-1} (x + 1)^{-1} \, dx \]
- \[ \int \tan(x) \, dx \]
Problem

*Use the logarithm function and its properties to evaluate the derivative of*

\[ f(x) = \frac{x^2(x - 4)^3}{(x^2 + 1)^4}. \]