§6.1–Inverse Functions

Tom Lewis

Fall Semester
2015

Outline

The inverse of a relation

One-to-one functions

Inverse functions

Finding inverse functions

The calculus of inverse functions
Definition
A relation in the plane is a set of ordered pairs \((a, b)\) in the plane.

Relations defined by equations
Typically we are interested in relations defined through equations.

The graph of a relation
We can graph the relation specified by \(x = y^3 + 3y^2 + 2y\) by simply plotting the ordered pairs.
Definition

The inverse of a relation $R$ in the plane is the relation $R^{-1}$ obtained by transposing (swapping) the $x$ and $y$-coordinates of each point of $R$. Thus $(b, a)$ is in $R^{-1}$ if and only if $(a, b)$ is in $R$.

Problem

• What is the inverse of the relation $R = \{(2, 3), (3, 3), (5, 2)\}$. Graph $R$ and its inverse; what is the geometric relationship between $R$ and its inverse?

• If a relation is given by the equation $x = y^3 + 3y^2 + 2y$, what equation gives the inverse relation?
The graph of a relation and its inverse

Here are the graphs of the relation \( x = y^3 + 3y^2 + 2y \) (in blue) and its inverse (in red). The point \( P \) and its inverse \( Q \) are shown.

Problem

Consider the following two examples of functions.

\[
f = \{ (1, -1), (2, 1), (3, 2), (4, 0) \}
\]

\[
g = \{ (1, 1), (2, 3), (3, 1), (4, 2) \}.
\]

- Graph the functions.
- In each case, find the inverse relations.
- Are the inverse relations functions?
- How are the domains and ranges of the functions and their inverse relations related?
- By what tests can we tell whether a function will have an inverse function?
Definition (One-to-one)

- A function $f$ with domain $D$ is called one-to-one if distinct elements of $D$ have distinct images. In other words,

$$f(s) = f(t) \text{ if and only if } s = t.$$  

- Said another way, a function is called one-to-one if it never takes on the same value more than once.

Problem

*Which of the specified functions is one-to-one?*

1. $f(x) = x^2$.
2. $f(x) = x/(1 + x)$
Theorem (Horizontal line test)

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Theorem (Increasing and decreasing)

If a function is either increasing or decreasing on an interval domain, then it is one-to-one.

Problem

Show that $f(x) = 2x + \sin(x)$ is one-to-one on $(-\infty, \infty)$. 
Problem

*Explain how to restrict the domain of the function* \( f(x) = x^2 \) *to make it one-to-one.*

**Definition (Inverse function)**

- Let \( f \) be one-to-one with domain \( A \) and range \( B \). The inverse function of \( f \), denoted by \( f^{-1} \), has domain \( B \) and range \( A \).
- \( f^{-1} \) maps \( y \) to \( x \) if and only if \( f \) maps \( x \) to \( y \).
- Equivalently, for any \( y \in B \),

\[
 f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y
\]
Theorem (Cancellation equations)

Let $f$ be one-to-one with domain $A$ and range $B$.

- $f^{-1}(f(x)) = x$ for each $x \in A$.
- $f(f^{-1}(y)) = y$ for each $y \in B$.

Problem

Let $f(x) = \sqrt{x - 4}$ on the interval $[4, \infty)$. Find $f^{-1}(x)$. 
Problem

Let \( f(x) = x^2 + 1 \) on the interval \([0, \infty)\). Show that \( f \) is invertible and find \( f^{-1} \).

Theorem

If \( f \) is a one-to-one, differentiable function with inverse function \( f^{-1} \), if \((a, b)\) is on the graph of \( f \), and if \( f'(a) \neq 0 \), then \( f^{-1} \) is differentiable at \( b \) and

\[
(f^{-1})'(b) = \frac{1}{f'(a)}
\]
Problem

Let \( f(x) = x^3 + 3x \). Find \( (f^{-1})'(4) \).

---

Problem

Let \( f(x) = 2x + \sin(x) \). Find \( (f^{-1})'(4\pi) \).