§4.2–The Definite Integral

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Spring Semester
2014

Outline

Why do we need to re-think area?
Riemann sums
The Riemann integral
Evaluating integrals
Properties of integrals
A more robust approach

- We will want to define area for functions which are discontinuous. It turns out that right-hand rule, the left-hand rule and the midpoint rule are not sufficient for this task.
- We will want to extend the idea of “area” to functions that take on both positive and negative values. In fact we want to define the signed area of the curve $y = f(x)$ on the interval $[a, b]$.

Signed area of $A$ is $4/3$; signed area of $B$ is $-4/3$; signed area of $C$ is $20/3$. Thus

$$\text{Total signed area} = \frac{20}{3}.$$
Problem (Handout)

Find the Riemann sum for the function \( f(x) = -x^2 + x + 2 \) for the partition \( P = \{0, 1, 2.5, 3.5, 4\} \) and the sample points \( \{.5, 2, 3.1, 4\} \).

Draw the corresponding system of rectangles.

Riemann sums

Let \( f \) be a bounded function on the interval \([a, b]\).

- Let \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \) be a partition of \([a, b]\).
  Let \( P = \{x_0, x_1, x_2, \ldots, x_n\} \).

- Let \( \Delta x_i = x_i - x_{i-1} \) denote the length of the \( i \)th subinterval, \([x_{i-1}, x_i]\). We will let \( \|P\| \) the length of the longest subinterval, called the norm of \( P \).

- For each \( i \), select \( x_i^* \in [x_{i-1}, x_i] \). The set \( \{x_1^*, x_2^*, \ldots, x_n^*\} \) is called the set of sample points.

- The Riemann sum based on this data is

\[
\sum_{i=1}^{n} f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n.
\]

Each summand corresponds to the signed area of a rectangle.
Definition (The Riemann Integral)

If \( f \) is bounded on the interval \([a, b]\), then the definite integral (or Riemann integral) of \( f \) from \( a \) to \( b \) is the number

\[
\int_a^b f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^n f(x^*_i) \Delta x_i,
\]

provided this limit exists. If the limit exists, then we say that \( f \) is integrable on \([a, b]\).

Corollary

If \( f \) is integrable on \([a, b]\), then the right-hand rule, the left-hand rule, and the midpoint rule will all converge to \( \int_a^b f(x) \, dx \), the definite integral of \( f \) on \([a, b]\).

Signed area

When \( f \) is integrable on \([a, b]\), then

\[
\int_a^b f(x) \, dx = \text{signed area trapped between } f \text{ and the } x\text{-axis}
\]

Theorem

- If \( f \) is continuous on \([a, b]\), then \( f \) is integrable on \([a, b]\), that is, \( \int_a^b f(x) \, dx \) exists.
- If \( f \) has only a finite number of jump discontinuities on \([a, b]\), then \( f \) is integrable on \([a, b]\), that is, \( \int_a^b f(x) \, dx \) exists.
**Problem**

*In each case, evaluate the integral by graphing the function and interpreting the integral as a signed area.*

1. \( \int_{0}^{3} (2 - x) \, dx \)
2. \( \int_{0}^{3} (4 + \sqrt{9 - x^2}) \, dx \)
3. \( \int_{-2}^{3} |x - 1| \, dx \)

**Theorem (Properties of sums)**

*Let \( c \) be a constant.*

1. \( \sum_{i=1}^{n} c = cn \)
2. \( \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \)
3. \( \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \)
4. \( \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \)
Theorem (Properties of integrals)

Let $c$ be a constant. Suppose that all of the following integrals exist. Then

1. $\int_{a}^{b} c \, dx = c(b - a)$

2. $\int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$

3. $\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx$

4. $\int_{a}^{b} (f(x) - g(x)) \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx$

Problem

Evaluate $\int_{-2}^{2} (3\sqrt{4 - x^2} - 5|x| + 2x - 5) \, dx$ using properties of integrals and geometry.
Theorem (The Splitting Property)

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

The splitting property is useful when your function is defined by cases.

Problem

Let

\[ f(x) = \begin{cases} 
3 - x & x \leq 1 \\
3x - 1 & x > 1 
\end{cases} \]

Graph \( f \) and evaluate \( \int_{-1}^{4} f(x) \, dx \).
**Definition**

Let $f$ be integrable on $[a, b]$. We define

$$\int_{b}^{a} f(x) \, dx = -\int_{a}^{b} f(x) \, dx.$$

**Theorem**

*The splitting formula,*

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$

*is valid for all orderings of $a$, $b$, and $c$.*

**Theorem (Comparison Properties of Integrals)**

- If $f(x) \geq 0$ for $x \in [a, b]$, then $\int_{a}^{b} f(x) \, dx \geq 0$.
- If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx$.
- If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then
  $$m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a).$$
Problem

Show that

\[
\frac{1}{2} \leq \int_0^1 \frac{1}{1 + x^2} \, dx \leq 1
\]