§4.1–Area and Distance

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Outline

An overview of the problem

The area under a “nice” function

Other evaluation schemes

Limits

The distance problem
Areas of plane figures

- Determining the area of a plane figure is an ancient problem.
- The areas of simple figures have been known from antiquity.
Archimedes and the circle
Archimedes showed that the area of a circle of radius $r$ is $\frac{1}{2}Cr$, where $C$ is the circumference of the circle. He showed this by approximating the circle from inside and outside with regular polygons.

Abstract “curvy” shapes
How can we find the area of a general plane figure? The argument for the circle suggests that we can approximate the area by simple plane figures, but how can we calculate the limit in this case?
The Area Problem
Calculate the area of the region bounded by $y = 1 + x^2$, the $x$-axis, $x = 0$, and $x = 3$.

Problem
Here is a picture of the right-hand rule for $n = 6$ subdivisions, denoted by $R_6$. Calculate the sum of the areas of the approximating rectangles.
**Problem**

The right-hand rule for $n = 12$ subdivisions. The sum of the areas of the rectangles is approximately 13.16.

![Graph showing the right-hand rule for 12 subdivisions]

**Problem**

The right-hand rule for $n = 24$ subdivisions. The sum of the areas of the rectangles is approximately 12.57.

![Graph showing the right-hand rule for 24 subdivisions]
**Numerical Results**

In the table below you can see the value of $R_n$ for increasing values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$R_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>12.1355</td>
</tr>
<tr>
<td>200</td>
<td>12.0676</td>
</tr>
<tr>
<td>300</td>
<td>12.0451</td>
</tr>
<tr>
<td>400</td>
<td>12.0338</td>
</tr>
<tr>
<td>500</td>
<td>12.027</td>
</tr>
<tr>
<td>600</td>
<td>12.0225</td>
</tr>
<tr>
<td>700</td>
<td>12.0193</td>
</tr>
<tr>
<td>800</td>
<td>12.0169</td>
</tr>
<tr>
<td>900</td>
<td>12.015</td>
</tr>
<tr>
<td>1000</td>
<td>12.0135</td>
</tr>
</tbody>
</table>

**Definition**

Given a sequence of numbers $a_1, a_2, a_3, \ldots, a_n$, define

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$ 

The letter $i$ is called the index of summation.

**Example**

We have

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$
Definition (The right-hand rule)

- Let \( f \) be a continuous, positive function on an interval \([a, b]\).
  Let \( n \geq 1 \) be an integer.
- Subdivide the interval \([a, b]\) into \( n \) equal pieces and let
  \[
  a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b
  \]
denote the resulting partition.
- \( \Delta x = \frac{(b - a)}{n} \) denotes the common width of the subintervals.
- The right-hand rule for this data is
  \[
  R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x
  = \sum_{i=1}^{n} f(x_i)\Delta x
  \]

Note
For computing purposes,
\[
R_n = \Delta x(f(x_1) + f(x_2) + \cdots + f(x_n))
\]
is usually the best formula.
The left-hand rule, $L_6$

![Graph showing the left-hand rule](image1)

The midpoint rule, $M_6$

![Graph showing the midpoint rule](image2)
Problem

1. Compute $L_6$ and $M_6$ for $f(x) = 1 + x^2$ on the interval $[0, 3]$.
2. How do $R_6$, $L_6$ and $M_6$ compare the actual area?

Theorem

Let $f$ be a continuous, positive function on the interval $[a, b]$. The area under the curve $y = f(x)$ on the interval $[a, b]$ can be expressed as

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x.$$ 

Note

In this theorem, the same result holds with $L_n$ or $M_n$ in place of $R_n$. 
Problem

Express the area under the graph of \( f(x) = x^2 + 2x + 3 \) on the interval \([1, 5]\) as a limit of right-hand rules.

Problem

The following corresponds to the limit of right-hand rules for a function \( f \) on an interval \( I \):

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( 6 + 9 \frac{i}{n} \right) \sin \left( 2 + 3 \frac{i}{n} \right) \frac{1}{n}.
\]

What are \( f \) and \( I \)?
Problem
A man runs on a track and each second his velocity \( (v) \) is recorded. The results of this experiment are listed below.

<table>
<thead>
<tr>
<th>( t ) (sec.)</th>
<th>0</th>
<th>.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (m/sec)</td>
<td>10.0</td>
<td>9.8</td>
<td>9.6</td>
<td>9.4</td>
<td>9.2</td>
<td>9.0</td>
<td>8.8</td>
<td>8.6</td>
<td>8.4</td>
<td>8.2</td>
<td>8.0</td>
</tr>
</tbody>
</table>

Approximately how far did the man run over this 10-second interval?

Theorem
Let a particle move on a line with velocity \( v(t) \). If \( v(t) \) is positive and continuous on the time interval \([a, b]\), then the total distance traveled by the particle over the time interval \([a, b]\) is the area under the velocity curve on the interval \([a, b]\).
Problem

The velocity of a particle is given by \( v(t) = 3 + 2t \) on the time interval \([1, 4]\). What is the total distance traveled by the particle?

Problem

The velocity of a particle is given by \( v(t) = \sqrt{4 - t^2} \) on the time interval \([0, 2]\). What is the total distance traveled by the particle?