Appendix A: Numbers, Inequalities, and Absolute Values

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Outline

Types of numbers

Notation for intervals

Inequalities

Absolute value
A hierarchy of numbers

**Whole numbers** 1, 2, 3, . . .; these are the natural counting numbers.

**Integers** . . . , −3, −2, −1, 0, 1, 2, 3, . . .; these are the whole numbers together with their additive inverses and 0. This is a useful abstraction of the whole numbers.

**Rational numbers** −3/2, 111/53, .77777 . . . These are numbers that can be represented in the form $p/q$ where $p$ and $q$ are integers. For example,

**Irrational numbers** $\sqrt{2}$, $\pi$, $\sqrt{5}$. These are the real numbers which are not rational; their decimal expansions do not terminate nor do they repeat.

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**Problem**

*Classify each of the following numbers in its lowest possible number type.*

- −6
- $3.134567676767 \ldots$
- $\frac{18}{5}$
- $\frac{\frac{4}{5} + \frac{2}{5}}{1 + \sqrt{3}}$
- $\sqrt{2}$
- $\sqrt{4 + \sqrt{12}}$
The real numbers as a set

The real numbers are **totally ordered**. This means that if we are given any two distinct real numbers \( x \) and \( y \), then they can be compared; thus, either

\[
 x > y \quad \text{or} \quad x < y.
\]

We can visualize the real numbers therefore in a (familiar) number line.

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Subsets

- A subset of real numbers is any collection of real numbers.
- By far the most important subsets of real numbers are the **intervals**.
  - From a geometrical viewpoint, the intervals correspond to **connected segments** within the real line.
  - The endpoints may or not be contained in an interval. An interval can have infinite length.
We have two preferred notations: interval and set-builder notation.

Problem

Sketch the intervals given by $(-\infty, 2]$ and $\{x : -3 \leq x < 8\}$.

Problem

Consider the inequality:

$$2x + 5 < 4x + 8.$$

- What does it mean to solve the inequality?
- What kind of answer do we expect?
- How do we solve the inequality?
Solving inequalities

We can solve an inequality just as we would an equality with one exception: multiplication by a negative number. If $a < b$ and $c < 0$, then $ca > cb$.

Problem

1. Solve $3x + 12 < 6x - 18$ for $x$.

2. Find the set of all $x$ such that

   $3 - 5x > -3$ \ and \ $3 - 5x < 18$.

3. Find the set of all $x$ such that

   $3x - 4 > 8$ \ or \ $3x - 4 < -8$.

4. Find the set of all $t$ such that $t^2 + 3t > 28$.  
Definition (Absolute value)

Given a real number $x$, the absolute value of $x$, denoted by $|x|$, is the distance from the 0 (the origin) to $x$. We can think of $|x|$ as the length of the number $x$.

Problem

*How can you make a simple calculator return the absolute value of an input?*
**Definition (Absolute value)**

We can formulate a precise definition of $|x|$ as follows:

$$
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}.
$$

**Problem**

*In each case, describe the indicated set as an interval or union of intervals:*

- $|x| \leq 3$.
- $|x| > 2$.
- $|x| \leq -1$. 
Problem

1. Solve $|x| = 8$.
2. Solve $|x| < 2$ and $|x| \geq 5$.
3. Solve $|x - 3| < 1$. What is the geometric meaning of this set?
4. Find $a$ and $b$ such that the solution set of $|x - a| < b$ is the interval $(-1, 7)$.
5. Solve $|2 - 3x| \geq 8$.
6. Solve the equation $|x + 2| = |3x - 1|$.

Some properties of absolute value

Let $a$ and $b$ be real numbers.

- $\sqrt{a^2} = |a|$ \\
- $|ab| = |a||b|$ \\
- $|a/b| = |a|/|b|$, provided that $b \neq 0$ \\
- $|a + b| \leq |a| + |b|$ (triangle inequality)
Problem

Suppose that $|x - 3| < .001$ and $|y - 5| < .002$. Use the triangle inequality to give an upper bound for $|(x + y) - 8|$.