

Practice Test #4

This practice test consists of the following problems

- §3.6: 16, 36
- §3.7: 2, 6, 8
- §3.8: 21, 27
- §3.9: 4, 16, 28, 34

§3.6, Problem 16 Differentiating both sides with respect to x yields:

$$\frac{1}{2}(x+y)^{-1/2} \left(1 + \frac{dy}{dx}\right) = 2xy^2 + 2x^2y \frac{dy}{dx}.$$

By distributing the term $\frac{1}{2}(x+y)^{-1/2}$, this can be re-written as

$$\frac{1}{2}(x+y)^{-1/2} + \frac{1}{2}(x+y)^{-1/2} \frac{dy}{dx} = 2xy^2 + 2x^2y \frac{dy}{dx}.$$

Next we collect the terms containing $\frac{dy}{dx}$ on the left and factor, yielding

$$\left(\frac{1}{2}(x+y)^{-1/2} - 2x^2y\right) \frac{dy}{dx} = 2xy^2 - \frac{1}{2}(x+y)^{-1/2}.$$

We can now solve for $\frac{dy}{dx}$, obtaining

$$\frac{dy}{dx} = \frac{2xy^2 - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} - 2x^2y}.$$

This could be simplified further.

§3.6, Problem 36 Note that a is a constant; thus,

$$4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad \text{hence} \quad \frac{dy}{dx} = \frac{-x^3}{y^3}.$$

We now take another derivative, being attentive to the quotient rule.

$$\frac{d^2y}{dx^2} = \frac{y^3(-3x^2) - (-x^3)(3y^2) \frac{dy}{dx}}{y^6}$$

We can resolve the double-negative in the numerator and substitute $-x^3/y^3$ in place of $\frac{dy}{dx}$ on the right, yielding

$$\frac{d^2y}{dx^2} = \frac{y^3(-3x^2) + (x^3)(3y^2) \frac{(-x^3)}{y^3}}{y^6}$$

To clear the compound fraction, let us multiply the numerator and denominator by y^3 .

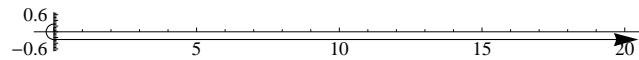
$$\frac{d^2y}{dx^2} = \frac{-3y^6x^2 - 3x^6y^2}{y^9} = \frac{(-3x^2y^2)(y^4 + x^4)}{y^9} = \frac{(-3x^2)(y^4 + x^4)}{y^7}$$

Since $x^4 + y^4 = a^4$, we can neatly express the second derivative as

$$\frac{d^2y}{dx^2} = \frac{-3x^2a^4}{y^7}$$

§3.7, Problem 2 Here we go!

1. $v(t) = f'(t) = .04t^3 - .12t^2 = .04t^2(t - 3)$.
2. $v(3) = 0$
3. The particle is at rest when $v(t) = 0$, that is, $.04t^2(t - 3) = 0$. The solutions are $t = 0$ and $t = 3$.
4. If we make a sign chart for the velocity function, we see that $v(t) < 0$ (moving to the left) on the time intervals $(-\infty, 0)$ and $(0, 3)$ and $v(t) > 0$ (moving to the right) on the time interval $(3, \infty)$.
5. The particle moves to the left on the time interval $(0, 3)$ seconds and then turns to the right on the time interval $(3, 8)$. Since $f(3) - f(0) = -.27$ and $f(8) - f(3) = 20.75$, we find that the total distance traveled is $.27 + 20.75 = 21.02$ feet.
6. This is the best that I can do:



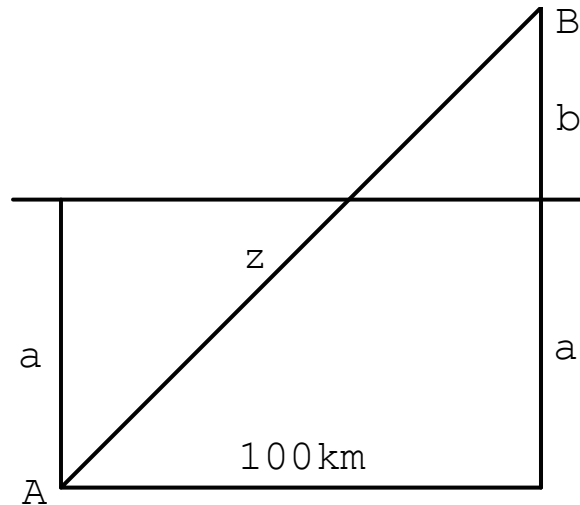
7. $a(t) = v'(t) = f''(t) = -0.24t + 0.12t^2 = 0.12t(t - 2)$.
8. Skip
9. When we stack the sign charts for the velocity and acceleration functions, we see that the particle is slowing down on the intervals $(-\infty, 0)$ and $(2, 3)$ and that the particle is speeding up on the intervals $(0, 2)$ and $(3, +\infty)$.

§3.7, Problem 6 We will do part (a) only, since it is the same idea for part (b) as well. Make a graph of the first derivative (the velocity). It will be parabolic shaped with its vertex at $t = 2$ and roots at $t = 1$ and $t = 3$. Thus $v(t) > 0$ on $[0, 1)$ and $(3, 4]$ and $v(t) < 0$ on $(1, 3)$. The graph of the second derivative (the acceleration) will be roughly a straight line with t -intercept at $t = 2$. Thus $a(t) > 0$ on the interval $(2, 4]$ and $a(t) < 0$ on $[0, 2)$. Thus we may conclude that the particle is speeding up on the intervals $(1, 2)$ and $(3, 4]$ and the particle is slowing down on the intervals $[0, 1)$ and $(2, 3)$.

§3.7, Problem 8 If $s(t) = 5t + 3t^2$, then it follows that $v(t) = 5 + 6t$.

1. $v(2) = 5 + 6(2) = 17\text{m/s}$.
2. We need to solve $v(t) = 35$, that is, $5 + 6t = 35$. The solution is $t = 5$ seconds.

§3.8, Problem 21 Let A and B mark the locations of the two ships at some generic time. Let a and b denote the distance that each ship has traveled since noon. Let z denote the distance between the ships. Here is a picture:



We know that $\frac{da}{dt} = 35\text{km/h}$ and $\frac{db}{dt} = 25\text{km/h}$. We want $\frac{dz}{dt}$. This means that we need to establish a relationship between a , b , and z . By adding two line segments to the picture, we see, by the Pythagorean theorem, that

$$z^2 = (a + b)^2 + 100^2.$$

Taking derivatives, yields

$$2z \frac{dz}{dt} = 2(a + b) \left(\frac{da}{dt} + \frac{db}{dt} \right) \quad \text{or} \quad \frac{dz}{dt} = \frac{(a + b) \left(\frac{da}{dt} + \frac{db}{dt} \right)}{z}.$$

The specific time in question is 4:00pm. At that time, $a = (4)(35) = 140\text{km}$ and $b = (4)(25) = 100\text{km}$. To solve for z , we note that

$$z^2 = (140 + 100)^2 + (100)^2 = 67600 \quad \text{hence} \quad z = 260$$

Finally,

$$\frac{dz}{dt} = \frac{(140 + 100)(35 + 25)}{260} = \frac{720}{13} \approx 55.385 \text{ km/h}.$$

§3.8, Problem 27 Let V , r , and h denote the volume, radius, and height of the cone. In this problem, we are given that $\frac{dV}{dt} = 30$ cubic feet per minute and we are asked to find $\frac{dh}{dt}$. Thus we are seeking a relationship between V and h . In general, the volume of a cone is given by $\frac{1}{3}\pi r^2 h$. In this problem, however, we know that the diameter of the cone is always equal to the height of the cone; thus, $2r = h$ or $r = h/2$. Thus, we can write the volume as

$$V = \frac{1}{3}\pi(h/2)^2 h = \frac{\pi}{12}h^3.$$

Differentiating with respect to t , reveals

$$\frac{dV}{dt} = \frac{3\pi}{12}h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{4 \frac{dV}{dt}}{\pi h^2}.$$

At the instant that the pile is 10 feet high, we see that

$$\frac{dh}{dt} = \frac{4(30)}{\pi(10)^2} = \frac{120}{100\pi} \approx .382 \text{ feet/min.}$$

§3.9, Problem 4 We have $f(x) = x^{3/4}$ and $a = 16$. Since $f(16) = (16)^{3/4} = 8$ and $f'(16) = \frac{3}{4(16)^{1/4}} = \frac{3}{8}$, we have

$$L(x) = 8 + \frac{3}{8}(x - 16).$$

§3.9, Problem 16 For $y = (x+1)^{-1}$, we have $y' = -1(1+x)^{-2}$. Thus for $x = 1$ and $dx = -0.01$, we have

$$dy = f'(x)dx = \frac{-1}{(1+1)^2}(-0.01) = \frac{.01}{4} = .0025.$$

§3.9, Problem 28 Since 99.8 is close to 100, clearly the square root of 99.8 is approximately 10. We can do a bit better with a linearization technique. Let $f(x) = x^{1/2}$ and $a = 100$. Since $f(a) = 10$ and $f'(a) = 1/(20)$, we see that the linearization of f centered at a is

$$L(x) = 10 + \frac{1}{20}(x - 100).$$

Since $f(x) \approx L(x)$ for x near a , it follows that

$$\sqrt{99.8} \approx L(99.8) = 10 + \frac{1}{20}(-0.2) = 10 - 0.01 = 9.99$$

The actual value of $\sqrt{99.8}$ is 9.989994995... Our approximation is quite good.

§3.9, Problem 34 After the paint is added, the radius of the hemisphere will be increased by .05cm. Thus the change in the volume of the hemisphere must be due to the paint. We can approximate the actual change in volume (ΔV) by the differential dV . Since the volume of a sphere is $\frac{4}{3}\pi r^3$, the volume of a hemisphere is $\frac{2}{3}\pi r^3$. The differential dV is related to dr by

$$dV = \frac{dV}{dr} dr = 2\pi r^2 dr.$$

For our problem $r = 50\text{m}$ and $dr = .05\text{cm} = .0005\text{m}$; thus,

$$dV = 2\pi(50)^2(.0005) = 2\pi(1.25) = 7.85\text{m}^3$$

Note that

$$\Delta V = \frac{2}{3}\pi(50.0005)^3 - \frac{2}{3}\pi(50)^3 = 7.85406\dots\text{m}^3.$$