Chapter 2

Applications of Matrix Theory: Markov Chains

2.1 Introduction to Markov chains

All that is required of probability theory is the simple notion that the probability of an outcome is the frequency with which that outcome will occur. Thus in the toss of a fair die, we would attach probability $1/6$ to occurrence of each face.

Problem 2.1.1. Ace Taxi divides a city into two zones: A and B. If a driver picks up a client in A, he will drop them off in A with probability .25 and drop them off in B with probability .75. If a driver picks up a client in zone B, he will drop them off in zone A with chance .4 and drop them off in zone B with chance .6. The company rules dictate that a driver must remain in a given zone for the next client.

Given that the driver starts in zone A compute the probabilities that the driver will be in zones A and B after two clients have been served. Given that the driver starts in zone B compute the probabilities that the driver will be in zones A and B after two clients have been served.

Solution. We can arrange the information in a transition diagram. See Figure 2.1. The transition diagram carries all of the probabilistic information concerning the movement of the taxis.

We can make tree diagrams to aid us in calculating the probabilities. Starting from A, we can move to A or to B. From each of those nodes, the

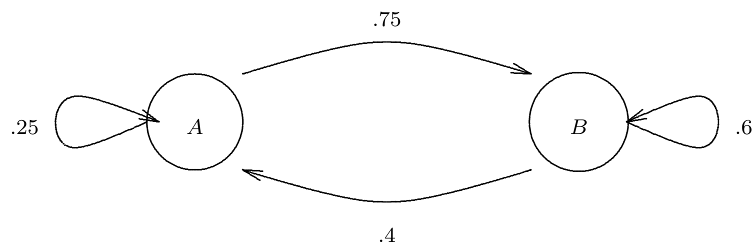

Figure 2.1: A transition diagram

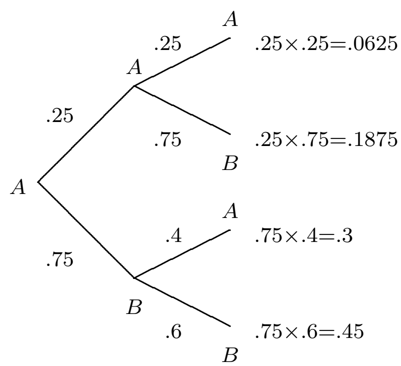

Figure 2.2: A tree diagram

same two alternatives are still available. See Figure 2.2. Each branch of the tree corresponds to a distinct outcome of the experiment. The probability of that outcome is given by multiplying the probabilities along the branches. Given that the taxi starts in zone A, the taxi will be in zone A after two steps if either of the outcomes AAA or ABA occurs. Accordingly the chance that the taxi is in zone A after two steps is $.0625+.3 = .3625$. Likewise given that the taxi begins in zone A, the taxi will be in zone B after two steps with chance $.1875 + .45 = .6375$. Notice that these alternatives add to 1.

A similar calculation would show that given that the taxi starts in zone B, the taxi will be in zone A after two steps with chance .34 and in zone B after two steps with chance .66. We will leave it to the reader to provide the necessary tree diagram for this calculation.

Definition 2.1.1. Let $S = \{s_1, s_2, \cdots, s_n\}$ be a set. Each element of S is called a *state* and S is called the *state space*. A *Markov chain* on S is a random process in discrete time whose value at any time is an element of S. The probability that the chain makes a transition from state s_i to state s_j depends only on s_i and s_j and not on any of the preceding states. In other words, the future depends only upon the present state of the chain and not on the past states of the chain.

Example 2.1.1. In the solution to Problem 2.1.1, the state space was $S = \{A, B\}$. The probabilities of transition were codified in the transition diagram. See Figure 2.1.

Our next theorem demonstrates how matrix theory can be utilized to compute probabilities.

Definition 2.1.2. Let the states of a Markov chain be labeled $1, 2, \ldots, n$. Given two states i and j, let

$$p_{ij} = \text{prob. of making a transition from state } i \text{ to } j$$

The transition matrix P for the Markov chain is the matrix whose (i, j) entry is p_{ij}, that is, $\mathsf{P} = ((p_{ij}))$.

Example 2.1.2. Recall the Markov chain from Problem 2.1.1. Let state A be labeled 1 and state B be labeled 2. The transition matrix is

$$\mathsf{P} = \begin{bmatrix} .25 & .75 \\ .4 & .6 \end{bmatrix}$$

Observe that each of the row sums of the transition matrix in Example 2.1.2 are 1. This is true in general.

Theorem 2.1.1. *If P is a transition matrix for a Markov chain, then each row of P sums to 1.*

Proof. Let the state space be $\{1, 2, \cdots, n\}$. Consider state i in the transition diagram for this Markov chain. Since the Markov chain must make a transition to one of the n states with probability 1, the sum total of all of the transition probabilities out of state i must add to 1. This implies that

$$p_{i1} + p_{i2} + \cdots + p_{in} = 1.$$

But this concludes our proof, since this is nothing more than the sum of the elements of the ith row of P. $\square$

Table 2.1: U-Haul Statistics

Pick-up	Delivery	Percent
1	1	30
1	2	20
1	3	50
2	1	40
2	2	20
2	3	40
3	1	50
3	2	20
3	3	30

Problem 2.1.2. A U-Haul franchise services three cities, labeled 1, 2 and 3. Table 2.1 shows the percentage of the time that a van is picked up in city i and delivered to city j. The movements of an individual moving van can be modeled by a Markov chain.

(1) Construct the transition diagram for this Markov chain.

(2) Find the transition matrix for this Markov chain.

(3) If a van starts in state 1, what is the chance that it is in state 3 after 2 steps?

Solution. The transition diagram is given in Figure 2.3

The transition matrix is

$$\mathsf{P} = \begin{bmatrix} .3 & .2 & .5 \\ .4 & .2 & .4 \\ .5 & .2 & .3 \end{bmatrix}$$

Finally, we can calculate the probability of starting in state 1 and being in state 3 after two steps by a tree diagram. The relevant branches are

$$1 \to 1 \to 3 : \text{chance } .15$$
$$1 \to 2 \to 3 : \text{chance } .08$$
$$1 \to 3 \to 3 : \text{chance } .15$$

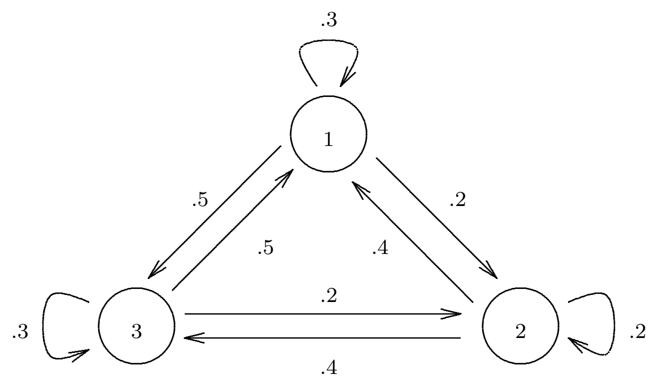

Figure 2.3: A transition diagram

Thus the probability in question is $.15 + .08 + .15 = .38$.

Multi-step transitions

Definition 2.1.3. Let P be a transition matrix for a Markov chain on a state space labeled $S = \{1, 2, \cdots, n\}$.

(1) For each integer $k \geq 1$, let $p_{ij}(k)$ denote the probability of a transition from i to j in k steps.

(2) For each integer $k \geq 1$, let $\mathsf{P}(k) = ((p_{ij}(k)))$ denote the matrix of k-step probabilities. $\mathsf{P}(k)$ is called the *k-step transition matrix.*

Theorem 2.1.2. *Let P be the transition matrix for a Markov chain. Then the k-step transition matrix $P(k)$ is given by*

$$\mathsf{P}(k) = \mathsf{P}^k$$

Proof. Let us assume that the state space is $S = \{1, 2, \ldots, n\}$. We will proceed by induction on n.

It is clear that $\mathsf{P}(1) = \mathsf{P}^1$, which proves the base case.

Let us assume that $\mathsf{P}(k) = \mathsf{P}^k$ for a positive integer k. Let $i, j \in S$. To compute $p_{ij}(k+1)$, we will partition this event according to the state of the Markov chain at time k. We can organize our work in a tree diagram, as in Figure 2.4. Thus

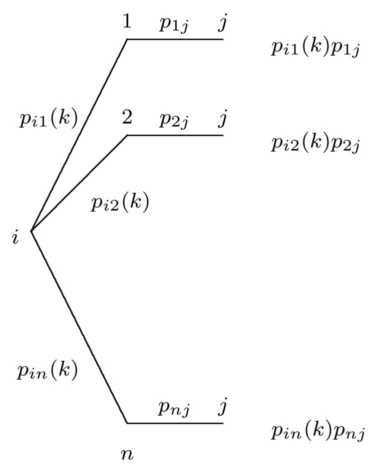

Figure 2.4: A multi-step tree

$$
\begin{aligned}
[\mathsf{P}(k+1)]_{ij} &= p_{ij}(k+1) \\
&= p_{i1}(k)p_{1j} + p_{i2}(k)p_{21} + \cdots + p_{in}(k)p_{nj} \\
&= [\mathsf{P}(k)\mathsf{P}]_{ij}
\end{aligned}
$$

which proves our claim. □

Example 2.1.3. Consider the transition matrix

$$\mathsf{P} = \begin{bmatrix} .3 & .7 \\ .6 & .4 \end{bmatrix}$$

Note that

$$\mathsf{P}^2 = \begin{bmatrix} .51 & .49 \\ .42 & .58 \end{bmatrix} \quad \text{and} \quad \mathsf{P}^8 = \begin{bmatrix} .46157 & .53843 \\ .46157 & .53843 \end{bmatrix}$$

We may conclude that: (1) the probability of making a transition from state 1 to state 2 in 2 steps is approximately .49; (2) the probability of making a transition from state 1 to state 2 in 8 steps is approximately .5348.

Example 2.1.4. Recall the transition matrix from Problem 2.1.2:

$$\mathsf{P} = \begin{bmatrix} .3 & .2 & .5 \\ .4 & .2 & .4 \\ .5 & .2 & .3 \end{bmatrix}$$

A simple calculation reveals that

$$\mathsf{P}(2) = \mathsf{P}^2 = \begin{bmatrix} .42 & .20 & .38 \\ .4 & .2 & .4 \\ .38 & .2 & .42 \end{bmatrix}$$

This shows that $p_{13}(2) = .38$, which is the same answer we arrived at above.

Exercises

1. A Markov chain has transition matrix

$$\mathsf{P} = \begin{bmatrix} .2 & .3 & .5 \\ .3 & .1 & .6 \\ .7 & .1 & .2 \end{bmatrix}$$

Assume that the chain begins in state 2. Find the probability that after two steps the chain will be in state 3. Calculate this in two ways: first, by a tree diagram; second, by calculating $\mathsf{P}(2)$.

2. A Markov chain has transition matrix

$$\begin{bmatrix} 0 & 1 \\ .3 & .7 \end{bmatrix}$$

Given that the Markov chain begins in state 1, what is the probability that the chain will be in state 1 after 8 steps?

3. In Greenville it is either raining or shining. If it is raining today, it will be raining tomorrow with chance .4 and shining with chance .6; if it is shining today, it will be raining tomorrow with chance .2 and shining with chance .8. Given that it is raining today, what are the respective chances for rain and shine in 6 days? in 20 days?

4. Each morning I eat one of three breakfast cereals: Cheerios, Special K, or Corn Flakes. I will stick with the cereal I ate on the previous morning 80% of the time, but when I do switch, I choose from among the other two with equal chance. Given that I am eating Cheerios today, what is the chance that I will be eating Corn Flakes for breakfast in 10 days? What am I most likely eating for breakfast in 10 days?

5. It is well known that the son of a Clemson graduate will go to Clemson with chance .9 and to Furman with chance .1. Likewise the son of a Furman man will go to Clemson with chance .3 and to Furman with chance .7. Given that a man is a Furman graduate, what is the chance that his great-great grandson will attend Furman?

6. On any given day my mood can be in one of three states: happy, sad, or morose. If I am happy on a given day, then on the next day I will be happy 80% of the time, sad 15% of the time, and morose 5% of the time. If I am sad on a given day, then on the next day I will be happy 50% of the time, sad 20% of the time, and morose 30% of the time. If I am morose on a given day, then on the next day I will be happy 40% of the time, sad 40% of the time, and morose 20% of the time. Given that I am sad today, find the probability that I will be in each of the three moods in 5 days.

7. Let

$$\mathsf{P} = \begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ .4 & .5 & .1 \end{bmatrix}$$

What is the influence of the 1 in row 1? Formulate a conjecture by examining the behavior of $\mathsf{P}(n)$ as n increases.

8. Find n-step transition matrices for

$$\mathsf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Interpret the associated Markov processes in each case.

9. The diffusion of a gas in a chamber can be modeled by a Markov chain. We will examine a very simple case of this model. Imagine that we have two buckets and three balls. We will distinguish one of the buckets by coloring it red, and we will begin our experiment with all of the balls in the red bucket. Each hour one of the balls is chosen at random, removed from its bucket, and placed in the other. Let the state of the Markov chain be the number of balls in the red bucket. Find the state space and transition matrix for this model. After 5 hours, what is the most likely state of the chain?

10. A mother has many children. The children want to know if they can stay out and play. The mother tells A either "yes" or "no", then A tells

B, then B tells C, and so on. Unfortunately these children are not good listeners: 10% of the time they hear the opposite of what of what they were told. What is the chance that the 10th child hears what his mother said?

Answers

1 $p_{23}(2) = .33$

2 $p_{11}(8) \approx .23082$

3 After 6 days, it will be raining with chance $\approx .25$ and shining with chance $\approx .75$. The same statistics hold after 20 days.

4 .32392; I am most likely eating Cheerios, but the margin is slim.

5 .132

6 happy: .69461; sad: .18940; and morose: .11599

8 $\mathsf{P}(n) = \mathsf{P}^n = \mathsf{P}$; and $\mathsf{Q}(n) = \mathsf{Q}^n = \mathsf{P}$ if n is even and Q if n is odd.

9 The state space is $S = \{0, 1, 2, 3\}$ according to the number of balls that can be in the red bucket. The transition matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

After 5 hours, it is most likely that the red bucket contains 1 ball; this occurs with chance $\approx .74$.

10 There are two states: let "yes" be labeled 1 and "no" be labeled 2. Let the state of the chain at time n be the message that the nth child hears. The transition matrix is

$$\mathsf{P} = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}.$$

The 10th child has chance $\approx .55369$ of hearing what his mother said.