8.8–Improper Integrals

Mark Woodard

Furman U

Fall 2010

Mark Woodard (Furman U)

§8.8–Improper Integrals

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Outline



2 Type I integrals: unbounded domains

3 Type II integrals: unbounded integrands

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Recall that the definite integral ∫_a^b f(x)dx is only defined for a bounded function f on a bounded domain [a, b]. Thus the following integrals are *improper*.

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- $\int_0^1 x^{-1/2} dx$ is improper because the integrand has a vertical asymptote at 0, which is in the domain of integration.

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- $\int_0^1 x^{-1/2} dx$ is improper because the integrand has a vertical asymptote at 0, which is in the domain of integration.
- $\int_0^\infty \frac{1}{x^{4/3} + x^{2/3}} dx$ is improper because the integrand is unbounded and the domain is unbounded.

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• When the domain of integration is unbounded, we must solve the problem by limits:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx,$$

whenever this limit exists.

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• Likewise
$$\int_{-\infty}^{b} f(x) dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) dx.$$

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Problem Evaluate $\int_{1}^{\infty} x^{-2} dx$

Evaluate
$$\int_{1}^{\infty} x^{-2} dx$$

Solution

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Solution

• Let b > 1 and observe that

$$\int_{1}^{b} x^{-2} dx = -x^{-1} \Big|_{1}^{b} = 1 - \frac{1}{b}.$$

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Solution

• Let b > 1 and observe that

$$\int_{1}^{b} x^{-2} dx = -x^{-1} \Big|_{1}^{b} = 1 - \frac{1}{b}.$$

• Since
$$\lim_{b o\infty}\left(1-b^{-1}
ight)=1$$
, we say

$$\int_1^\infty x^{-2} dx = 1.$$

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• Let b > 1. Then

$$\int_{1}^{b} x^{-1} dx = \ln(x) \Big|_{1}^{b} = \ln(b) - \ln(1) = \ln(b)$$

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• Since $\lim_{b\to\infty} \ln(b) = +\infty$, we say that the integral diverges. However we will sometimes write

$$\int_1^\infty x^{-1} dx = +\infty$$

to indicate that the integral diverges in this particular way.

Evaluate the following integrals:

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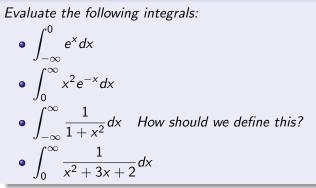
•
$$\int_{-\infty}^{0} e^{x} dx$$

•
$$\int_{0}^{\infty} x^{2} e^{-x} dx$$

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Evaluate the following integrals: • $\int_{-\infty}^{0} e^{x} dx$ • $\int_{0}^{\infty} x^{2} e^{-x} dx$ • $\int_{-\infty}^{\infty} \frac{1}{1+x^{2}} dx$ How should we define this?

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Type II integrals: unbounded integrands

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• If f is unbounded as $x \to a^+$, then we define

$$\int_a^b f(x) dx = \lim_{c \to a^+} \int_c^b f(x) dx,$$

provided the limit exists.

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provided the limit exists.

• If f is unbounded as $x \to b^-$, then we define

$$\int_a^b f(x) dx = \lim_{c \to b^-} \int_a^c f(x) dx,$$

provided the limit exists.

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Evaluate
$$\int_0^1 x^{-1/2} dx$$
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Solution

• Let 0 < c < 1 and observe that

$$\int_{c}^{1} x^{-1/2} dx = 2x^{1/2} \Big|_{c}^{1} = 2 - 2\sqrt{c}$$

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• Since
$$\lim_{c\to 0^+} (2-2\sqrt{c}) = 2$$
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Evaluate
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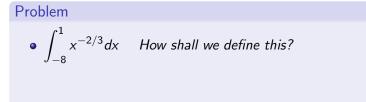
Solution

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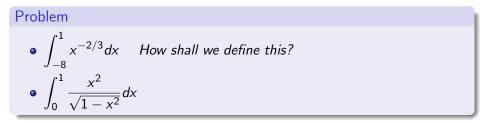
$$\int_{c}^{1} x^{-1/2} dx = 2x^{1/2} \Big|_{c}^{1} = 2 - 2\sqrt{c}$$

• Since
$$\lim_{c\to 0^+} (2-2\sqrt{c}) = 2$$
.
• Thus $\int_0^1 x^{-1/2} dx = 2$.

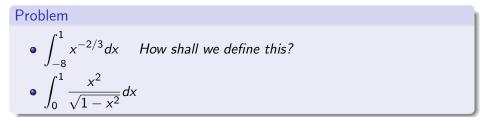
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Here is an integral of Type I and Type II. Evaluate

$$\int_{0}^{\infty} \frac{1}{x^{4/3} + x^{2/3}} dx$$

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