§8.8–Improper Integrals

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Outline

1. An overview

2. Type I integrals: unbounded domains

3. Type II integrals: unbounded integrands
An overview

What makes an integral improper?

Recall that the definite integral
\[ \int_a^b f(x) \, dx \]
is only defined for a bounded function \( f \) on a bounded domain \([a, b]\).

Thus the following integrals are improper:

\[ \int_0^\infty e^{-x} \, dx \text{ is improper because the domain, } [0, \infty), \text{ is unbounded.} \]

\[ \int_0^1 x^{-1/2} \, dx \text{ is improper because the integrand has a vertical asymptote at } 0, \text{ which is in the domain of integration.} \]

\[ \int_0^{\infty} \frac{1}{x^{4/3} + x^{2/3}} \, dx \text{ is improper because the integrand is unbounded and the domain is unbounded.} \]
An overview

What makes an integral improper?

- Recall that the definite integral $\int_a^b f(x)\,dx$ is only defined for a *bounded* function $f$ on a *bounded* domain $[a, b]$. Thus the following integrals are *improper*:

  \[ \int_0^\infty e^{-x}\,dx \text{ is improper because the domain, } [0, \infty), \text{ is unbounded.} \]

  \[ \int_0^1 \frac{x^{-1/2}}{2}\,dx \text{ is improper because the integrand has a vertical asymptote at } 0, \text{ which is in the domain of integration.} \]

  \[ \int_0^\infty \left( \frac{1}{x^{4/3}} + \frac{1}{x^{2/3}} \right)\,dx \text{ is improper because the integrand is unbounded and the domain is unbounded.} \]
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  - \( \int_0^{\infty} e^{-x}\,dx \) is improper because the domain, \([0, \infty)\), is unbounded.
  - \( \int_0^1 x^{-1/2}\,dx \) is improper because the integrand has a vertical asymptote at 0, which is in the domain of integration.
  - \( \int_0^{\infty} \frac{1}{x^{4/3} + x^{2/3}}\,dx \) is improper because the integrand is unbounded and the domain is unbounded.
Type I: unbounded domains

When the domain of integration is unbounded, we must solve the problem by limits:

\[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx, \]

whenever this limit exists.

When this limit exists, we say that the improper integral \( \int_{a}^{\infty} f(x) \, dx \) converges. When this limit does not exist, we say that the integral diverges.

Likewise \( \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \).
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- Likewise \( \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx. \)
Problem

Evaluate \( \int_{1}^{\infty} x^{-2} \, dx \)
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Solution

Let \( b > 1 \) and observe that

\[
\int_{1}^{b} x^{-2} \, dx = \left[ -x^{-1} \right]_{1}^{b} = 1 - b^{-1}.
\]

Since \( \lim_{b \to \infty} (1 - b^{-1}) = 1 \), we say

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\int_{1}^{\infty} x^{-2} \, dx = 1.
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- Let \( b > 1 \) and observe that

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Solution

Let \( b > 1 \). Then
\[
\int_1^{b} x^{-1} \, dx = \ln(x) \bigg|_1^b = \ln(b) - \ln(1) = \ln(b).
\]
Since \( \lim_{b \to \infty} \ln(b) = +\infty \), we say that the integral diverges.
However we will sometimes write \( \int_1^{\infty} x^{-1} \, dx = +\infty \) to indicate that the integral diverges in this particular way.
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**Solution**

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- \( \int_{-\infty}^{0} e^x \, dx \)
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1. $\int_{-\infty}^{0} e^x \, dx$
2. $\int_{0}^{\infty} x^2 e^{-x} \, dx$
3. $\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$
Problem

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- $\int_{-\infty}^{0} e^x \, dx$
- $\int_{0}^{\infty} x^2 e^{-x} \, dx$
- $\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$  \textit{How should we define this?}
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2. \( \int_{0}^{\infty} x^2 e^{-x} \, dx \)
3. \( \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx \)  \text{How should we define this?}
4. \( \int_{-\infty}^{\infty} \frac{1}{x^2 + 3x + 2} \, dx \)
Type II integrals: unbounded integrands

If \( f \) is unbounded as \( x \to a^+ \), then we define
\[
\int_a^b f(x) \, dx = \lim_{c \to a^+} \int_a^c f(x) \, dx,
\]
provided the limit exists.

If \( f \) is unbounded as \( x \to b^- \), then we define
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\int_a^b f(x) \, dx = \lim_{c \to b^-} \int_c^b f(x) \, dx,
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Problem

Evaluate \( \int_{0}^{1} x^{-1/2} \, dx \).
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Solution

Let \( 0 < c < 1 \) and observe that

\[
\int_1^c x^{-1/2} \, dx = 2x^{1/2} \bigg|_{1}^{c} = 2 - 2\sqrt{c}.
\]

Since \( \lim_{c \to 0^+} (2 - 2\sqrt{c}) = 2 \),

Thus \( \int_0^1 x^{-1/2} \, dx = 2 \).
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Evaluate $\int_0^1 x^{-1/2} \, dx$.

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- Since $\lim_{c \to 0^+} (2 - 2\sqrt{c}) = 2$.

- Thus $\int_0^1 x^{-1/2} \, dx = 2$. 
Problem

Problem

Here is an integral of Type I and Type II. Evaluate

\[ \int_{0}^{\infty} \frac{1}{x^{4/3} + x^{2/3}} \, dx \]
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\[ \int_{-8}^{1} x^{-2/3} \, dx \quad \text{How shall we define this?} \]
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\[ \int_{0}^{1} \frac{x^2}{\sqrt{1 - x^2}} \, dx \]
Problem

- $\int_{-8}^{1} x^{-2/3} \, dx$  \textit{How shall we define this?}
- $\int_{0}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx$

Problem

\textit{Here is an integral of Type I and Type II. Evaluate} $\int_{0}^{\infty} \frac{1}{x^{4/3} + x^{2/3}} \, dx$