§8.8–Improper Integrals

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Outline

What makes an integral improper?

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- Recall that the definite integral $\int_a^b f(x)dx$ is only defined for a *bounded* function f on a *bounded* domain $[a, b]$. Thus the following integrals are *improper*:

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- $\displaystyle\int_0^1 x^{-1/2}dx$ is improper because the integrand has a vertical asymptote at 0, which is in the domain of integration.

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- $\displaystyle\int_0^1 x^{-1/2}dx$ is improper because the integrand has a vertical asymptote at 0, which is in the domain of integration.
- $\displaystyle\int_0^\infty \frac{1}{x^{4/3} + x^{2/3}}dx$ is improper because the integrand is unbounded and the domain is unbounded.

Type I: unbounded domains

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- When the domain of integration is unbounded, we must solve the problem by limits:

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- When this limit exists, we say that the improper integral $\int_a^\infty f(x)dx$ *converges*. When this limit does not exist, we say that the integral *diverges*.

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  whenever this limit exists.
- When this limit exists, we say that the improper integral $\int_a^{\infty} f(x)dx$ *converges*. When this limit does not exist, we say that the integral *diverges*.
- Likewise $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$

Problem

Evaluate $\int_1^\infty x^{-2}dx$

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Solution

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Solution

- *Let* $b > 1$ *and observe that*

$$\int_1^b x^{-2} dx = -x^{-1}\Big|_1^b = 1 - \frac{1}{b}.$$

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- *Let* $b > 1$ *and observe that*

$$\int_1^b x^{-2}dx = -x^{-1}\Big|_1^b = 1 - \frac{1}{b}.$$

- *Since* $\lim_{b\to\infty}\left(1 - b^{-1}\right) = 1$, *we say*

$$\int_1^\infty x^{-2}dx = 1.$$

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- *Let* $b > 1$. *Then*

$$\int_1^b x^{-1}dx = \ln(x)\Big|_1^b = \ln(b) - \ln(1) = \ln(b).$$

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Solution

- *Let* $b > 1$. *Then*

$$\int_1^b x^{-1} dx = \ln(x)\Big|_1^b = \ln(b) - \ln(1) = \ln(b).$$

- *Since* $\lim_{b\to\infty} \ln(b) = +\infty$, *we say that the integral diverges. However we will sometimes write*

$$\int_1^{\infty} x^{-1} dx = +\infty$$

to indicate that the integral diverges in this particular way.

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- $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$ *How should we define this?*

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- $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+3x+2}\,dx$

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- If f is unbounded as $x \to a^+$, then we define

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provided the limit exists.

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provided the limit exists.

- If f is unbounded as $x \to b^-$, then we define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx,$$

provided the limit exists.

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Solution

- *Let* $0 < c < 1$ *and observe that*

$$\int_c^1 x^{-1/2} dx = 2x^{1/2}\Big|_c^1 = 2 - 2\sqrt{c}.$$

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Solution

- *Let* $0 < c < 1$ *and observe that*

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- *Since* $\lim_{c \to 0^+} \left(2 - 2\sqrt{c}\right) = 2.$

Problem

Evaluate $\int_0^1 x^{-1/2}dx$.

Solution

- *Let* $0 < c < 1$ *and observe that*

$$\int_c^1 x^{-1/2}dx = 2x^{1/2}\Big|_c^1 = 2 - 2\sqrt{c}.$$

- *Since* $\lim_{c\to 0^+}\left(2 - 2\sqrt{c}\right) = 2$.
- *Thus* $\int_0^1 x^{-1/2}dx = 2$.

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Problem

Here is an integral of Type I and Type II. Evaluate $\int_{0}^{\infty} \frac{1}{x^{4/3}+x^{2/3}} dx$