§12.8 –Power Series

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Outline

Problem

For each x, consider the series

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots$$

There are two natural questions to ask about this series:

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- *For which values of x, does this converge?*

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There are two natural questions to ask about this series:

- *For which values of x, does this converge?*
- *Is this a recognizable function?*

Definition

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- A *power series* is an infinite series of the form

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  The numbers $\{c_n\}$ are called the *coefficients* of the power series.
- Given a number a, we define the power series centered at a by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a)^1 + c_2(x-a)^2 + \cdots, \quad x \in \mathbb{R}.$$

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The radius of convergence

The number R is called the radius of convergence. We can think of these three cases accordingly: $R = 0$, $0 < R < \infty$, or $R = \infty$.

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- *If $R = \infty$, then the interval of convergence is $(-\infty, +\infty)$.*
- *If $0 < R < \infty$, then the interval of convergence can be any one of the following:*

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R].$$

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- $\displaystyle\sum_{n=1}^{\infty} n!x^n$